Ranker's

BLUEPRINT

JEE ADVANCED MATHS

New Problems

~ with ~

Innovative Solutions

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FOUNDATIONS OF TRADITION & LOGIC

DEDICATED TO
MY GRANDMOTHERS

Late Mintra Devi

⊗

Late Vishambhari Thakur

“Sa Vidya Ya Vimuktaye”

AXIOM: KNOWLEDGE AS THE ULTIMATE VARIABLE OF LIBERATION

This work is a tribute to the **Sanatan philosophy of life**: a realization that we are but temporary vessels for eternal truths. Through the wisdom of my grandmothers, I learned that the pursuit of mathematical excellence is a **Dharma**—an algorithmic duty to decode the universe.

Preface

The contemporary landscape of the JEE Advanced has undergone a definitive paradigm shift. We have transitioned from an era of rote algebraic manipulation to one defined by structural depth and computational logic. This evolution is no coincidence; the global explosion in **Data Science** and algorithmic reasoning has profoundly influenced the IITs' approach to problem-setting. Modern examination patterns now prioritize pattern recognition and multi-layered intuition—the very hallmarks of a data-driven intellect.

Following the success of *Thin Book with Fat Ideas*, this volume, **Blueprint for Rankers**, is engineered for this new reality. It eschews the traditional "encyclopedic" approach in favor of **Fat Ideas**—concepts that are minimalist in their statement but gargantuan in their implications. Every problem herein has been meticulously curated to align with the updated JEE syllabus, ensuring that the aspirant's focus remains strictly on the rigors of the current examination cycle.

The solutions provided are not merely answers; they are **innovations**. They emphasize "First-Principle Thinking" to dismantle problems that appear insurmountable, offering the reader a strategic advantage in an increasingly competitive field.

I owe a debt of gratitude to the vibrant community at **Namma Ganit** and my family for their unwavering support. Their collective intellectual curiosity and the overwhelming response to my previous works have been the primary catalysts for this manuscript.

To the aspirant: Mathematics at this level is a game of strategy as much as skill. May this blueprint serve as your definitive guide to mastering the rules.

Kunal Thakur
Author

The JEE Advanced Ideal Game: A Biological Strategy for Peak Intellectual Performance

The Namma Ganit Protocol for High-Stakes Success

"The exam is a 6-hour game of cognitive endurance. To win, your biology must be as sharp as your logic. None of us are eternal; play your best game today."

I. Neuro-Nutrition: Fueling the Biological Processor

In the "Ideal Game," food is not for taste; it is for **Cognitive Stability**. Avoid the "Sugar Crash" that leads to silly mistakes in Paper II.

- **Low Glycemic Index Protocol:** Consume slow-release carbohydrates (complex grains) to ensure a steady supply of glucose to the brain during long problem-solving sessions.
- **Omega-3 Fatty Acids:** Incorporate healthy fats to maintain the integrity of neuronal membranes, facilitating faster electrical signaling.
- **Anti-Inflammatory Intake:** Prioritize antioxidants to reduce "brain fog" caused by systemic inflammation.

II. Physical Conditioning: The Kinetic Foundation

A stagnant body leads to a stagnant mind. Use movement to flush the system and reset the "Logic Circuits."

- **Postural Alignment:** Engage in core and back strengthening to endure hours of seated concentration without physical fatigue.
- **Aerobic Flush:** Use high-intensity movement to increase **Cerebral Blood Flow**, delivering fresh oxygen to the prefrontal cortex after a grueling mock test.
- **Neuromuscular Coordination:** Sports like table tennis or badminton sharpen the hand-eye coordination required for rapid-fire bubbling and calculation.

III. Systemic Recovery: Memory Consolidation through Sleep

Sleep is where the "Rank" is actually made. It is the phase where information moves from short-term "RAM" to long-term "Hard Drive" storage.

- **Glymphatic Cleaning:** Achieve deep sleep to allow the brain to clear metabolic byproducts that accumulate during intense conceptual learning.
- **REM Dominance:** Prioritize the 7–9 hour window to maximize Rapid Eye Movement sleep, which is critical for complex pattern recognition and creative problem-solving.

IV. The Meditative Anchor: Mental Equilibrium

The JEE is won by those who stay calm when the paper is "tough." Meditation is the training for that exact moment.

- **Amygdala Regulation:** Use breath-work to shrink the brain's fear center, preventing "panic-blanking" during unfamiliar questions.
- **Stoic Happiness:** Acknowledge that while the exam is a game, your worth is defined by your effort. Being happy and present reduces the performance anxiety that hampers deep thinking.

Execute the Protocol. Master the Game. Stay Happy.

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The Ranker's Blueprint

"Inhale the chaos. Exhale the solution."

SINGLE CORRECT CHOICE QUESTIONS

Q.1

The value of $\left\|\int_0^1\left((1-x^{2026})^{\frac{1}{2026}}-x\right)^2 dx\right\|$ is:	
(A)	$\frac{2}{3}$
(B)	$\frac{1}{2}$
(C)	$\frac{1}{3}$
(D)	$\frac{3}{2}$

Q.2

Consider the Parabola $P: y^2 = 4x$. Let $M(x_1,\ y_1)$ be a variable point lying on the curve $y^2 = x$ such that M is the midpoint of chord of Parabola P. The combined equation of line joining origin (0, 0) to endpoints of this chord represents the asymptotes of a variable Hyperbola $H(x_1)$. Let $e(x_1)$ denotes eccentricity of $H(x_1)$. Then which is **TRUE**?	
(A)	$\frac{de}{dx_1}(\text{at } x_1 = 2) = 0$
(B)	$\frac{de}{dx_1}(\text{at } x_1 = 8) = 0$
(C)	$\frac{dx_1}{de}(\text{at } x_1 = 2) = 0$
(D)	$\frac{dx_1}{de}(\text{at } x_1 = 8) = 0$

Q.3	Let A be the set of all complex numbers z with Im (z) $\neq 0$ such that the matrix $M=\begin{bmatrix} z & 1 \\ -1 & z+1 \end{bmatrix}$ has purely real determinant. Let z_0 be a fixed complex number and r be a positive real parameter satisfying $\lvert z_0 \rvert^2=1-2r^2$. Consider a specific element $\alpha \in A$ such that $\lvert \alpha - z_0 \rvert = r$ and $\left\lvert \frac{1}{\overline{\alpha}} - z_0 \right\rvert = 2r$. Then the value of $\lvert 8\lvert\alpha\rvert^2 + 16\mathrm{Re}(\alpha)+10 \rvert$ is:	
	(A)	4
	(B)	5
	(C)	6
	(D)	7

Q.4	Let f(n) denotes the number of ordered triplets of non-negative integers (a, b, c) such that $2a + b + c = n$. Then the value of $\left\lvert \sum_{n=0}^{\infty} \frac{f(n)}{2^n} \right\rvert$ is:	
	(A)	$\frac{8}{3}$
	(B)	$\frac{11}{3}$
	(C)	$\frac{16}{3}$
	(D)	$\frac{17}{3}$

Q.5	The number of distinct arrangements of **SIMPLE** such that none of the letters **S, I, M, P** appear in their original position:	
	(A)	326
	(B)	362
	(C)	322
	(D)	366

Q.6	Consider the set S of four 3×3 real matrices: $$S=\left\{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix},\begin{bmatrix}\cos\frac{\pi}{3}&-\sin\frac{\pi}{3}&0\\\sin\frac{\pi}{3}&\cos\frac{\pi}{3}&0\\0&0&1\end{bmatrix},\begin{bmatrix}3&0&0\\0&1&0\\0&0&1\end{bmatrix}\right\}$$ Let P be the set of all possible finite products of one or more matrices from set S. Then which of the following matrices cannot be an element of set P?
(A)	$\begin{bmatrix}9&0&0\\0&1&0\\0&0&1\end{bmatrix}$
(B)	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
(C)	$\begin{bmatrix}\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\0&0&1\end{bmatrix}$
(D)	$\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}$

Q.7	Consider a Tetrahedron T defined by origin O = (0,0,0), A = (a,0,0), B = (0,b,0), C = (0,0,c). Let d_1, d_2, d_3 be the shortest distances between three pairs of opposite edges of this tetrahedron. If $S=\{(a,b,c)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\frac{1}{d_1^2}+\frac{1}{d_2^2}+\frac{1}{d_3^2}=\frac{3}{4}\}$. Then the number of elements in set S is:
(A)	1
(B)	2
(C)	3
(D)	4

Q.8

Consider two curves C_1 and C_2 given by: $C_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + x - y^2 + y = 0\}$ $C_2 = \{(x,y) \in \mathbb{R}^2 \mid ax^2 + (a+1)xy + 2ax + y^2 + 2y = 0\}$ $C_1 \cap C_2 = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \in C_1 \text{ and } (x,y) \in C_2\}$ **Note:** $\lvert X\rvert$ denotes number of elements in set X	
(A)	$\exists$ some $a \in \mathbb{R}$ such that $\lvert C_1 \cap C_2 \rvert > 2026$
(B)	$\exists$ some $a \in \mathbb{R}$ such that $\lvert C_1 \cap C_2 \rvert = 2026$
(C)	$\max \lvert C_1 \cap C_2 \rvert = 4$ for any $a \in \mathbb{R}$
(D)	$\max \lvert C_1 \cap C_2 \rvert = 8$ for any $a \in \mathbb{R}$

Q9.

$S = \left\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid \sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)\right\}$ The area bounded by the locus of points in set 'S' is/are: (Here, the inverse trigonometric function $\sin^{-1} x$ assume values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$	
(A)	$2 + \frac{\pi}{2}$
(B)	$2 + \frac{\pi}{4}$
(C)	π
(D)	$\frac{\pi}{2}$

Q10.

Let A(a, 1), B(2, b) and C(3, c) be vertices of a triangle Δ ABC such that $\lvert \overrightarrow{BA} - t\overrightarrow{BC} \rvert \geq \lvert \overrightarrow{AC} \rvert$ for all $t \in \mathbb{R}$. $S = \{(a,b,c) \in \mathbb{Z}^3 \mid a,b,c \in \{1,2,3,4,5\}\}$. Then the number of elements in set S is/are:	
(A)	5
(B)	9
(C)	14
(D)	29

Q11.

Consider a 2×2 random matrix M:

$M = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. The entries x_i $(i = 1, 2, 3, 4)$ are independent and identically distributed random variable. Each x_i takes values from set {0, 1} with the following probabilities: $P(x_i = 1) = \frac{2}{3}$ and $P(x_i = 0) = \frac{1}{3}$

Then the variance of determinant of M is/are

(A)	$\frac{20}{41}$
(B)	$\frac{40}{41}$
(C)	$\frac{40}{81}$
(D)	$\frac{20}{81}$

Q12.

$\left| \frac{\int_0^{100} e^x \{x\} dx}{\int_0^1 e^x \{x\} dx} \right|$ is

(A)	100
(B)	e^{90}
(C)	$\frac{e^{100} - 1}{e - 1}$
(D)	$\frac{e^{100} + 1}{e + 1}$

Q13.

Let $S = \left\{ \begin{bmatrix} a & b\sqrt{5} \\ -b\sqrt{5} & a \end{bmatrix} : a, b \in \mathbb{Z} \right\}$ be the set of all 2×2 matrices with $\mathbb{Z}$ being the set of integers.

Let $G = \{A \in S : A^{-1} \in S\}$. Which of the following is (are) TRUE?

(A)	The number of elements in G is 2
(B)	The number of elements in G is infinite
(C)	If $A \in G$ then A^{2026} is not in G
(D)	$\sum_{B \in G} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Q14.	Let L_1 and L_2 be two skew lines in $\mathbb{R}^3$ defined by sets: $L_1 = \{\vec{r} \in \mathbb{R}^3 : \vec{r} = \vec{a} + \lambda\vec{c}, \lambda \in \mathbb{R}\}$ $L_2 = \{\vec{r} \in \mathbb{R}^3 : \vec{r} = \vec{b} + \mu\vec{d}, \mu \in \mathbb{R}\}$ Where $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = 2\hat{i} + \hat{j}$, $\vec{c} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{d} = \hat{i} + \hat{k}$. Define two planes P_1 and P_2 as follows: $P_1 = \{\vec{r} \in \mathbb{R}^3 : (\vec{r} - \vec{a})\cdot(\vec{c} \times (\vec{c} \times \vec{d})) = 0\}$ $P_2 = \{\vec{r} \in \mathbb{R}^3 : (\vec{r} - \vec{b})\cdot(\vec{d} \times (\vec{c} \times \vec{d})) = 0\}$ Let $S = P_1 \cap P_2$ be the line of intersection of planes P_1 and P_2 and $W = \left\{\vec{r} \in \mathbb{R}^3 : \min_{\vec{p} \in L_1} \lvert \vec{r} - \vec{p} \rvert = \min_{\vec{q} \in L_2} \lvert \vec{r} - \vec{q} \rvert\right\}$ Then which of the following is **FALSE**?
(A)	$L_1 \cap S = \{(1,1,0)\}$
(B)	$L_2 \cap S = \{(1,0,1)\}$
(C)	$\left(\frac{1}{2}, 1, \frac{1}{2}\right) \in W \cap S$
(D)	$\min \lvert \vec{p} - \vec{q} \rvert = \sqrt{2}$ where $\vec{p} \in L_1$ and $\vec{q} \in L_2$.

Q15.	Let S be the set of all points (x, y) in cartesian plane $\mathbb{R}^2$. Define three subsets of S representing lines as follows: $L_1 = \{(x, y) \in S : 2x - y = 7\}$ $L_2 = \{(x, y) \in S : 5x + y = 42\}$ $L_3 = \{(x, y) \in S : x + y = 14\}$ Let $R_i : S \to S$ be function representing the reflection of a point across the line L_i. Consider the following two set of points on the coordinate axes: $P = \{(x, y) \in S : y = 0\}$ $Q = \{(x, y) \in S : x = 0\}$ If there exists pair of points $X \in P$ and $Y \in Q$ such that the composition $R_1(R_2(R_3(X))) = Y$. Then sum of all possible value of t^2 where t is distance between points X and Y.
(A)	98
(B)	260
(C)	368
(D)	86

Q16.

Let S be the set of all circles in plane $\mathbb{R}^2$. If the lengths of tangents drawn from points $P_1(3,4)$, $P_2(3,7)$ and $P_3(7,4)$ to a member $F \in S$ are 1, 2 and 3 units respectively.

Let $F \in S$ be defined by set of points $F = \{(x,y) \in \mathbb{R}^2 : (x-h)^2 + (y-k)^2 = r^2\}$ where (h, k) is centre and r is radius. Then the ordered triplet (number of such F, h + k, r) can be:

(A)	(1, 9, 2)
(B)	(1, 9, 1)
(C)	(2, 7, 2)
(D)	(2, 9, 1)

Q17.

$$S = \left\{ z : \operatorname{Re}\left(\frac{z-1}{z+1}\right) = 0, z \in \mathbb{C} \setminus \{-1\} \right\}$$

$\max_{z \in S} |z^3 - z + 2|^2$ is

(Here, $\mathbb{C}$ is set of complex numbers and Re z denotes Real part of z)

(A)	17
(B)	15
(C)	13
(D)	11

Q18.

All 5-letter words formed using the letters of the word GANIT are arranged in an English dictionary and assigned serial numbers n = 1, 2, …, 120. Let W_n be the word at serial number n. The probability $P(W_n)$ of choosing a word W_n is defined by $P(W_n) = \frac{n}{2^n k}$ for some constant k.If $P(\text{GANIT}) = \frac{27 \cdot 2^{\alpha}}{2^{\beta} - 61}$ where $\alpha, \beta \in \mathbb{N}$. Then $\alpha + \beta$ is

(A)	212
(B)	211
(C)	210
(D)	209

Q19.	Let F be a set of functions defined on $\mathbb{R}\setminus\{-1,-2\}$ as: $F=\left\{f_1(x)=\frac{x}{1+x}, f_2(x)=\frac{3x+1}{x+1}, f_3(x)=\frac{2x+1}{x+2}\right\}$ Let S be set of all functions generated by composition of exactly 4 functions from F. $S=\{g(x) \mid g(x)=(h_4 o h_3 o h_2 o h_1)(x), \text{where } h_i \in F \text{ for all i}\}$ Then which of the following elements DOES NOT belong to set S?
(A)	$\frac{3x+1}{10x+4}$
(B)	$\frac{19x+17}{28x+26}$
(C)	$\frac{28x+9}{26x+9}$
(D)	$\frac{2x+17}{x+15}$

Q20.	Let $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=4\hat{i}+2\hat{j}+7\hat{k}$ be position vectors of two points in $\mathbb{R}^3$. Consider the sets S and T defined by: $S=\left\{\vec{r}\in\mathbb{R}^3 : |\vec{r}-\vec{a}|^2-|\vec{r}-\vec{b}|^2=50\right\}$ $T=\left\{\vec{r}\in\mathbb{R}^3 : |\vec{r}-\vec{b}|^2-|\vec{r}-\vec{a}|^2=50\right\}$ Let L_1 be the line of intersection of the locus S and the plane $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=0$. Let L_2 be the line of intersection of the locus T and the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=5$. Then the minimum distance between line L_1 and L_2 is:
(A)	5
(B)	10
(C)	$\frac{50}{\sqrt{29}}$
(D)	$10\sqrt{2}$

Q21.

Let A(n, a, d) denote a set of first n terms in an arithmetic progression with first term a and common difference d. Let $\sigma^2(S)$ denotes the variance of the set S. Consider the following three sets of observations: $S_1 = A(50, 2025, 4)$ $S_2 = A(50, 2026, 2)$ $S_3 = A(50, 2027, 3)$ If $\sigma_1^2, \sigma_2^2, \sigma_3^2$ represent the variances of $S_1, S_2\ S_3$ respectively, then which of the following relations is **TRUE**?	
(A)	$\sigma_1^2 < \sigma_2^2 < \sigma_3^2$
(B)	$\sigma_3^2 < \sigma_2^2 < \sigma_1^2$
(C)	$\sigma_2^2 < \sigma_3^2 < \sigma_1^2$
(D)	$\sigma_2^2 < \sigma_1^2 < \sigma_3^2$

Q22.

Let $f(x) = \lvert x^8 - x^7 + x^2 - x + 15 \rvert e^x + (\lvert x \rvert + a)^2 e^{(5-\lvert x \rvert)^2}, x \in \mathbb{R}$ Suppose $S = \{a \in \mathbb{R} \mid$ f(x) is DIFFERENTIABLE at all x $\in \mathbb{R}\}$. Then $\sum_{a \in S} a$ is	
(A)	5
(B)	10
(C)	$\frac{1}{5}$
(D)	$\frac{1}{2}$

Q23.

The value of $\left\lvert \dfrac{\int_0^1 \frac{dt}{\sqrt{1-t^4}}}{\int_0^1 \frac{dt}{\sqrt{1+t^4}}} \right\rvert^2$ Evaluates to :	
(A)	1
(B)	2
(C)	3
(D)	4

Q24.

Let P be the parabola $y^2=4x$. Let N_P be the locus of points $(x, y) \in \mathbb{R}^2$ such that the number of distinct real normals that can be drawn from (x, y) to the parabola P is strictly less than 3. If the area of region S defined by: $S = \{(x, y) \in \mathbb{R}^2 : y^2 \leq 4x\} \cap N_p$ is A. Then the area A is	
(A)	$\frac{272\sqrt{2}}{15}$
(B)	$\frac{352\sqrt{2}}{15}$
(C)	$\frac{272\sqrt{3}}{15}$
(D)	$\frac{352\sqrt{3}}{15}$

Q25.

Let S be the set of all column vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $a, b, c \in \{0,1\}$. Consider the following system of linear equations in x, y and z: $(a+1)x + y + z = a$; $x + (b+1)y + z = b$; $x + y + (c+1)z = c$. The number of elements in set S such that the system has atleast one solution.	
(A)	4
(B)	5
(C)	7
(D)	8

Q26.		
	Let X_n be the number of heads obtained from n independent coin tosses, where the probability of getting a head in a single toss is p. Let P_n be the probability that X_n is an even number. Then which of the following is TRUE?	
	(A)	$\lim_{n\to\infty} P_n = \frac{1}{4}$ for $p = \frac{1}{2}$
	(B)	$\lim_{n\to\infty} P_n = \frac{1}{2}$ for $p = \frac{1}{4}$
	(C)	$\lim_{n\to\infty} P_n = \frac{1}{3}$ for $p = \frac{1}{2}$
	(D)	$\lim_{n\to\infty} P_n = \frac{1}{4}$ for $p = \frac{1}{4}$

Q27.		
	Let $S_{10}(x) = \sum_{r=1}^{10} \frac{\sin(rx)}{r}$ be defined for $x \in (0,\pi)$. Let the set of critical points of $S_{10}(x)$ be denoted by x_k for k=0, 1, 2, …., m where $S_{10}^{1}(x_K) = 0$ and the points are ordered such that $0 < x_0 < x_1 < x_2 < \ldots < x_m < \pi$. Then $\lim_{x\to x_0}\left(\frac{\sin(10x) - 2\sin^2(5x)\tan\left(\frac{x}{2}\right)}{\cos\left(\frac{11x}{2}\right)}\right)$ is	
	(A)	$\frac{x_0}{11\pi}$
	(B)	$\frac{11x_0}{\pi}$
	(C)	$\frac{22x_0}{\pi}$
	(D)	$\frac{x_0}{22\pi}$

Q28.

Let $f_1, f_2, \ldots, f_n$ be sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ defined for $n \in \mathbb{N}$ as follows: $f_1(x) = \sin(\pi x)$, $f_2(x) = (2x-1)\sin(\pi x)$ and $f_{n+2}(x) = (2x-1)f_{n+1}(x) - f_n(x)$ for $n \geq 1$ Let $I_n = \int_0^1 f_n(x)dx$ for $n = 1,2,3,\ldots$ If $S = \{I_n : n \in \mathbb{N}\}$ is set of all distinct values in the sequence $I_1, I_2, I_3, \ldots$ then the number of elements in set S is:	
(A)	5
(B)	7
(C)	9
(D)	Infinite

Q29.

Let S be a finite set of vectors in $\mathbb{R}^3$ (3-D space) containing at least two non-zero vectors $\vec{a}$ and $\vec{b}$ such that $\vec{a}$ is not scalar multiple of $\vec{b}$. If for every $\vec{u}, \vec{v}$ and $\vec{w} \in S$, the vector $\vec{u} \times (\vec{v} \times \vec{w}) \in S$. Then the minimum number of elements in set S is:	
(A)	9
(B)	5
(C)	7
(D)	3

Q30.

Let $f : [-8,8] \to \mathbb{R}$ be given by $f(x) = \begin{cases} \sqrt{16-(x+8)^2}, & -8 \leq x \leq -4 \\ x^2 - 16, & -4 \leq x \leq 4 \\ \sqrt{16-(x-8)^2}, & 4 \leq x \leq 8 \end{cases}$ Then $\min\left(\int_{-8}^{8} \lvert f(x) - k \rvert \, dx\right)$ is	
(A)	$8\pi + \frac{256}{3}$
(B)	$8\pi + \frac{25}{3}$
(C)	$8\pi + \frac{26}{3}$
(D)	$8\pi + \frac{56}{3}$

Q31.	Consider a 3×3 matrix M and a column vector X as: $M = \begin{bmatrix} 1/4 & 1/8 & 1/16 \\ 0 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}$ and $X = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ Then $\lim_{n\to\infty} M^n X$ is $(n \in \mathbb{N})$
(A)	$\begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}$
(B)	$\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$
(C)	$\begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$
(D)	$\begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix}$

Q32.	A family of lines F is represented by the equation $F: \frac{x}{t} + \frac{y}{1-t} = 1$ where $t \in (0,1)$ A curve C is the locus of points such that every member of F is tangent to curve at some point for x, y > 0. Then Area represented by curve C, y axis and x axis is (sq. units)
(A)	$\frac{1}{18}$
(B)	$\frac{1}{12}$
(C)	$\frac{1}{6}$
(D)	$\frac{1}{3}$

Q33.

Define T(A, B) to be the number of ways A chocolate can be distributed among B children when every child gets at least one chocolate (chocolates are identical). Then:	
(A)	$\sum_{i=3}^{12} T(i+1,3) = 285$
(B)	$\sum_{i=3}^{12} T(i+1,3) = 275$
(C)	$\sum_{i=3}^{10} T(11,i+1) = 252$
(D)	$\sum_{i=3}^{10} T(11,i+1) = 352$

Q34.

Let S be the set of all fourth-degree polynomials $P(x) = x^4 + ax^3 + bx^2 + cx + d$ with $a, b, c, d \in \mathbb{R}$, such that all roots of P(x) are real and positive. Let k be the largest real number satisfying the inequality $(b-a-c)^2 \geq kd$ for all polynomials $P(x) \in S$. The value of k is	
(A)	144
(B)	121
(C)	196
(D)	169

Q35.

Let A and B be 3 × 3 matrices defined as:

$$A = \begin{bmatrix} a & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & b & \frac{1}{5} \\ c & \frac{1}{6} & \frac{1}{7} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{8} & d & \frac{1}{9} \\ \frac{1}{10} & \frac{1}{11} & e \\ \frac{1}{12} & \frac{1}{13} & f \end{bmatrix}$$

Where the variables a, b, c, d, e, f $\in \{0,1\}$. If $\det(AB - BA) = \dfrac{\operatorname{tr}((AB-BA)^3)}{3}$.

The number of ordered pairs of matrix (A, B) is

(A)	45
(B)	48
(C)	54
(D)	64

Q36.

$S = \{f : \mathbb{R} \to \mathbb{R} \mid f(2x) - f(x) = x \text{ for all } x \in \mathbb{R}\}$ $T = \{f : \mathbb{R} \to \mathbb{R} \mid f(0) = 1\}$ $P = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is CONTINUOUS}\}$ Then which of the following is CORRECT? (Note: |X| denotes number of elements in set X)	
(A)	$\lvert S \cap T \rvert = 1$
(B)	$\lvert S \cap T \cap P \rvert = 1$
(C)	$\lvert S \cap P \rvert = 1$
(D)	$\lvert S \cup T \cup P \rvert = 2$

Q37.

Let f(k) denotes the real solution to the equation $x \ln(ex) = k$ for $k \geq 1$. Then $\lim_{k \to \infty} \left(\frac{f(k) \ln k}{k} \right)$ is:	
(A)	0
(B)	e
(C)	1
(D)	2e

Q38.

$\lim_{n \to \infty} \left((8n^3 + 4n^2 + n + 11)^{\frac{1}{3}} - (4n^2 + n + 9)^{\frac{1}{2}} \right)$ where $n \in \mathbb{N}$, is:	
(A)	$\frac{1}{6}$
(B)	$\frac{1}{12}$
(C)	$\frac{1}{18}$
(D)	0

Q39.

Let A_k be a 3×3 matrix given by: $A_k=\begin{bmatrix}1 & k & 1\\ 0 & 1 & 0\\ 0 & 1 & 0\end{bmatrix}$

Denote $S=\{(k,n)\in\mathbb{N}\times\mathbb{N}:(A_k)^n=A_{75}\}$. Let $M=\sum_{(k,n)\in S}A_k$. Then which of the following statement is CORRECT?

(A)	$\operatorname{tr}(\operatorname{adj}(M))=20$
(B)	$\det(M+\operatorname{adj}(M))=0$
(C)	Set S has exactly 5 elements.
(D)	$\prod_{(k,n)\in S}(A_k^{2026})=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 1\\ 0 & 0 & 1\end{bmatrix}$

Q40.

Let P_1 be the parabola given by equation $y = x^2$. Let P_2 be the parabola obtained by rotating the curve counter clockwise about the origin by the acute angle $\theta\in(0,\frac{\pi}{2})$. Let Ω denotes the bounded region enclosed by the intersection of the parabolas P_1 and P_2. If the area of region Ω is 9, then the value of expression $\frac{\cos\theta}{1-\sin\theta}$ is:

(A)	1
(B)	2
(C)	3
(D)	4

Q41.

Let $\alpha_n=(2+\sqrt{3})^n=I_n+f_n$ where $I_n\in\mathbb{N}$ and $f_n\in(0,1)$ for all $n\in\mathbb{N}$. Then $\lim_{n\to\infty}f_n$ is:

(A)	0
(B)	1
(C)	$\frac{1}{\sqrt{3}}$
(D)	$\frac{1}{2\sqrt{3}}$

MULTIPLE CORRECT CHOICE

Q42.

Let S be the set of real values of λ for which the system $\begin{cases}\lambda x+2y=5\\ 8x+\lambda y=10\end{cases}$ has no solution. Let $\omega\in S$ and consider the matrix $P=\begin{bmatrix}1&1&1\\2&\omega&3\\1&4&\omega\end{bmatrix}$. Let Q be a matrix such that PQ = $kI\,(k\neq 0)$. If q_{23} = -1 then which of the following is/are **TRUE**?	
(A)	k=7
(B)	$\lvert\operatorname{adj}(Q)\rvert=3^{12}$
(C)	If μ is value of λ for which system has infinite many solutions, $\lvert P\rvert+\mu=31$
(D)	$Q=\operatorname{adj}(P)$

Q43.

Let $f:\mathbb{R}\to(0,1)$ be a continuous function. Which of the following equation(s) must have at least one solution for x in (0, 1)?	
(A)	$\begin{vmatrix}\int_0^x f(t)dt & \sin(\pi x)\\ -f(x) & \pi\cos(\pi x)\end{vmatrix}=0$
(B)	$\begin{vmatrix}\int_0^x f(t)dt & x^{2026}\\ 1 & 1\end{vmatrix}=0$
(C)	$x^{2026}=\int_0^x f(t)\cos t\,dt$
(D)	$\begin{vmatrix}e^x & \int_0^x f(t)dt\\ 1 & 1\end{vmatrix}=1$

Q.44	Let $\vec{w}=\hat{i}+\hat{j}-2\hat{k}$. Let $\vec{u},\vec{v}$ be two non-zero vectors such that $\vec{u}\times\vec{v}=\vec{w}$ and $\vec{v}\times\vec{w}=\vec{u}$. Let $\vec{u}$ be represented as $\vec{u}=\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ where $\alpha,\beta,\gamma\in\mathbb{R}$. Which of the following is/are CORRECT?
(A)	$\lvert\vec{v}\rvert=1$
(B)	Volume of parallelopiped formed by coterminous vectors $\vec{u},\vec{v},\vec{w}$ is 6
(C)	$\max\lvert\alpha+\beta+\gamma\rvert=3\sqrt{2}$
(D)	$\max\lvert\alpha+\beta+\gamma\rvert=2\sqrt{3}$

Q45.	For every integer n, let a_n and b_n be real numbers. Let function $f:\mathbb{R}\to\mathbb{R}$ be given by: $f(x)=\begin{cases} a_n x^2+b_n\cos(\pi x) & \text{for } x\in[2n,2n+1) \\ \frac{1}{\pi}(a_n\sin(\pi x)+b_n x) & \text{for } x\in[2n+1,2n+2) \end{cases}$ If f is differentiable on $\mathbb{R}$ then which of the following hold(s) for all n?
(A)	$a_n+2b_n=0$
(B)	$a_{n-1}-b_n=\frac{1}{\pi}(a_{n-1}+b_{n-1})-1$
(C)	$a_n+b_n=0$
(D)	$a_n=\frac{b_n}{\pi}$

Q46.	Let $T_k=\tan^{-1}\left(\frac{2}{8k^2-4k-1}\right)$ for $k\in\mathbb{N}$, and let $S_n=\sum_{k=1}^{n}T_k, n\in\mathbb{N}$ and $S=\sum_{k=1}^{\infty}T_k$. Which of the following(s) is(are) CORRECT?
(A)	$S=\int_0^1\frac{dx}{1+x^2}$
(B)	$\tan(S_{10})=\frac{20}{21}$
(C)	$\lim_{n\to\infty}n(\tan(S-S_n))=\frac{1}{4}$
(D)	$\lim_{n\to\infty}n(\tan(S-S_n))=\frac{1}{12}$

Q47.		
	Let $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$. Consider the function defined on $[0,\alpha]$ by $g(x) = \sum_{k=1}^{n} k \sin\left(\frac{k\pi x}{\alpha}\right)$. Let J_n be defined by $J_n = \int_0^\alpha g(x)\sin\left(\frac{\pi x}{\alpha}\right)dx$. Then which of the following statements is/are TRUE?	
	(A)	$J_{10} = J_{20}$
	(B)	For $\alpha = \pi$, $J_{20} = \frac{\pi}{2}$
	(C)	$J_{10} < \alpha\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{10^2}\right)$
	(D)	$J_{20} > \alpha\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{20^2}\right)$

Q48.		
	Two fair coins S_1 and S_2 are tossed independently once. Let the events E and F and G be defined as follows: E: Head appears on S_1 F: Head appears on S_2 G: The same outcome (head or tail) appears on both S_1 and S_2 Then which of the following statements is CORRECT?	
	(A)	E and F are independent
	(B)	F and G are independent
	(C)	E and G are independent
	(D)	E, F, G are mutually independent

Q49.

Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors satisfying $\vec{a} = \vec{b} \times \vec{c} + 2\vec{b}$ where $\|\vec{b}\| = \|\vec{c}\| = 2$ and $\|\vec{a}\| \le 4$. Then the possible value(s) of $\|2\vec{a} + \vec{b} + \vec{c}\|$ is/are:	
(A)	8
(B)	16
(C)	12
(D)	20

Q50.

Let $A(K) = \begin{bmatrix} K & 1 \\ 0 & \frac{1}{K} \end{bmatrix}$ where $K \in \mathbb{R} \setminus \{0, \pm 1\}$. Let I be the 2×2 identity matrix and $\text{adj}(M)$ denote the adjoint of matrix M. Let $A^n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$ for $n \in \mathbb{N}$. Then which of the following statement(s) is/are **CORRECT**?	
(A)	There exists an invertible matrix P such that $PA(5)P^{-1} = \begin{bmatrix} 5 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$
(B)	$b_{10} = \dfrac{K^{11} - K^{-9}}{K^2 - 1}$
(C)	$\text{Trace}(A^{10} + \text{adj}(A^{10})) = 2^{11} + 2^{-9}$ for k=2
(D)	$\sum_{K=20}^{26} \|A(K)^{-1} + \text{adj } A(K)\| = 8$

Q51.

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \dfrac{4^x}{2 + 4^x}$. Then which of the following is/are TRUE?	
(A)	$\int_{1/4}^{3/4} f(x)dx = \dfrac{1}{4}$
(B)	$\int_0^1 x f''(x)dx = \dfrac{4\ln 2 - 3}{9}$
(C)	$\sum_{r=1}^{\infty} \dfrac{f'(r)}{\ln 4 (f(r))^2} = \dfrac{4}{3}$
(D)	$f''\left(\dfrac{1}{2}\right) = 0$

Q52.

$$S = \left\{ (t, L) \in \mathbb{R} \times \mathbb{R} : L = \lim_{n \to \infty} \frac{e\left(1 - \frac{1}{n}\right)^n - 1}{n^t} \right\}.$$ Then which of the following(s) is (are) TRUE?

(A)	$\left(-1, -\frac{1}{2}\right) \in S$
(B)	$\left(-\frac{1}{2}, -1\right) \in S$
(C)	$(1, 0) \in S$
(D)	$\left(2, \frac{1}{2}\right) \in S$

Q53.

Let P be the parabola $y^2 = 4x$. A family of circles F is such that every member is tangent to P at point M(1, 2). From a fixed point A(-1, 2), two tangents (not necessarily distinct) are drawn to variable circle $S \in F$, touching it at points T_1 and T_2. Let H(h, k) be the midpoint of segment T_1T_2 as the circle S varies within the family F. Then which of the following (s) is (are) possible values of (h, k)?

(A)	$(-1, 2)$
(B)	$(\sqrt{2}, 1)$
(C)	$(0, -1)$
(D)	$(1, 2)$

Q54.	
	Let $\mathbb{R}$ denotes the set of all real numbers. Let $z_1 = 3+4i$ and $z_2 = 9+4i$ be two complex numbers, where $i=\sqrt{-1}$. Let S be the set defined by $S=\{x+iy : x, y \in \mathbb{R} \text{ and } \lvert x+iy-z_1\rvert + \lvert x+iy-z_2\rvert = 10\}$. Let $P_1, P_2, \ldots, P_{2026}$ be 2026 distinct points on the locus of S such that coordinates (x_k, y_k) of point P_k is given by $x_k = 6+5\cos\left(\frac{\pi k}{1013}\right)$ and $y_k = 4+4\sin\left(\frac{\pi k}{1013}\right)$ for $k=1,2,\ldots,2026$. Then which of the following statements is (are) TRUE?

(A)	S is an ellipse with eccentricity $\frac{3}{5}$
(B)	S is an ellipse with 20π square units.
(C)	$\left\lvert \dfrac{\sum_{k=1}^{2026} x_k y_k}{\sum_{k=1}^{2026} x_k} \right\rvert = 4$
(D)	$\min_{(x,y)\in S} \lvert (x-6)+i(y-8)\rvert = 1$

Q55.	
	Let $\mathbb{R}^3$ denote 3-D space. Denote three subsets of $\mathbb{R}^3$ as S_1, S_2 and S_3 given by: $S_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x\hat{i}+y\hat{j}+z\hat{k} = \alpha\hat{i}, \alpha \in \mathbb{R} \right\}$ $S_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x\hat{i}+y\hat{j}+z\hat{k} = \beta\left(\frac{1}{2}\hat{i}+\frac{\sqrt{3}}{2}\hat{j}\right), \beta \in \mathbb{R} \right\}$ $S_3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x\hat{i}+y\hat{j}+z\hat{k} = 5\hat{i}+5\hat{j}+\gamma\hat{k}, \gamma \in [-4,4] \right\}$ Let $\vec{u} \in S_1, \vec{v} \in S_2, \vec{w} \in S_3$ such that sum of distances between then is defined as $S = \lvert\vec{u}-\vec{v}\rvert + \lvert\vec{v}-\vec{w}\rvert + \lvert\vec{w}-\vec{u}\rvert$. where Column matrix $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ is represented by $u\hat{i}+v\hat{j}+w\hat{k}$. Then which of the following statements is(are) **TRUE**?

(A)	$\min S = 5\sqrt{6}$ for all such $\vec{u}, \vec{v}, \vec{w}$
(B)	$\min S = 6\sqrt{5}$ for all such $\vec{u}, \vec{v}, \vec{w}$
(C)	There is exactly one triplet of $(\vec{u}, \vec{v}, \vec{w})$ for which global minimum of S is achieved.
(D)	There are two triplets of $(\vec{u}, \vec{v}, \vec{w})$ for which global minimum of S is achieved.

Q56.

Let α and β be the roots of the equation $x^2 - x - 1 = 0$.

Let a sequence T_m for any integer $m \geq 1$ be defined as: $T_m = \sum_{k=0}^{m} (\alpha^k + \beta^k)$.

Then which of the following(s) is (are) possible value of T_m for some $m \geq 1$?

(A)	29
(B)	75
(C)	198
(D)	321

Q57.

Let f, g: [0, 1] $\rightarrow$ [0, 1] be defined as:

$$f(x) = \frac{1}{2}x(x+1) \text{ and } g(x) = \frac{1}{2}x^2(x+1)$$

Let f^{-1} and g^{-1} denote the inverse function of f and g respectively. Consider the following regions in the x-y plane defined by:

$$R_1 = \{(x,y) |\ x \in [0,1], \min(f^{-1}, g^{-1}) \leq y \leq \max(f^{-1}, g^{-1})\}$$
$$R_2 = \{(x,y) |\ x \in [0,1], \min(f, g) \leq y \leq \max(f, g)\}$$
$$R_3 = \{(x,y) |\ x \in [0,1], \min(f, f^{-1}) \leq y \leq \max(f, f^{-1})\}$$
$$R_4 = \{(x,y) |\ x \in [0,1], \min(g, g^{-1}) \leq y \leq \max(g, g^{-1})\}$$

If A_i represents the area of Region R_i, then which of the following statement(s) is (are) **CORRECT**?

(A)	$A_4 > A_3 > A_1$
(B)	$A_1 = \frac{3}{8}$
(C)	$A_2 = \frac{1}{8}$
(D)	$A_4 = \frac{5}{12}$

Q58.	For two events A and B associated with a random experiment let, $P(A)=\frac{3}{5}$ and $P(B)=\frac{2}{3}$. Then which of the following(s) is (are) **CORRECT**?
(A)	$P(A\cap\bar{B})\le\frac{1}{3}$
(B)	$P(A\cup B)\ge\frac{2}{3}$
(C)	$\frac{4}{15}\le P(A\cap B)\le\frac{3}{5}$
(D)	$\frac{1}{10}\le P(\bar{A}\mid B)\le\frac{3}{5}$

Q59.	Let S be the set of all ordered pairs (a, b) $\in\mathbb{R}^2$ such that the limit $\lim_{x\to 0}\left(\frac{a\sin x - x(1+b\cos x)}{x^3}\right)=1$ exists and is finite. If (a, b) $\in S$ then which of the following statements is/are **TRUE**?
(A)	The set S contains exactly one element.
(B)	All points (a, b) $\in S$ lies on the hyperbola $x^2 - y^2 = 10$.
(C)	$\lim_{x\to 0}\frac{\ln(1+ax)-\ln(1+bx)}{x}=1$
(D)	$(4, 2\sqrt{2})\in S$

Q60.	Let $\mathbb{C}$ denote the set of complex numbers. Consider the set A defined by: $A=\{z\in\mathbb{C}:z^{10}=1\}$ A new set B is constructed from A using the following definition: $B=\{w\in\mathbb{C}:w=3\text{Re}(z)+4i\ \text{Im}(z),\forall z\in A\}$ The elements of set B lie on a conic curve T. Let F_1 and F_2 denote the foci of T, with F_1 having a positive imaginary part. (**Note:** $i=\sqrt{-1}$ and $\text{Re}(z),\text{Im}(z)$ represent Real and imaginary part of complex number z respectively). Then which of the following(s) is (are) **TRUE**?
(A)	Eccentricity of T is $\frac{\sqrt{7}}{4}$
(B)	$\sum_{w\in B}\|w\|^2=125$
(C)	$\sum_{w\in B}d(w,F_1)=40$ where $d(w, F_1)$ is distance of $w\in B$ from focus F_1.
(D)	Let $\text{Area}(P_X)$ be the area of the polygon formed by the elements of set X. Then $\frac{\text{Area}(P_B)}{\text{Area}(P_A)}=12$

Q61.	Let $\hat{u}$ and $\hat{v}$ be two unit vectors in $\mathbb{R}^3$ with angle $\theta\in[0,\pi]$ between them. Let $\vec{w}$ be a vector satisfying the equation: $\vec{w}+(\vec{w}\times\hat{u})=\hat{v}$ .Which of the following statements is (are) **CORRECT**?
(A)	Let $V=\|[\vec{w}\hat{u}\hat{v}]\|$ denotes volumes of the parallelopiped formed by the vectors $\vec{w},\hat{u}$ and $\hat{v}$. If $S=\{V:\theta\in[0,\pi]\}$, then the maximum element of the set S is $\frac{1}{2}$.
(B)	For any fixed pair of unit vectors $\hat{u}$ and $\hat{v}$, the number of vectors $\vec{w}$ satisfying the given equation is exactly 2.
(C)	$\|\vec{w}\|\in\left[\frac{1}{\sqrt{2}},1\right]$
(D)	There always exists some scalars t, r $\in\mathbb{R}$ such that $\vec{w}=t\hat{u}+r(\hat{u}\times\hat{v})$ holds true for all such $\hat{u},\hat{v}$ and $\vec{w}$ given by above equation.

Q62.

Let $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ be a non-zero 3×1 matrix, and let M be a 3×3 matrix given by:

$$M = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Let S be the set of 3×1 matrices B such that $B^T B = 1$. For any $B \in S$, let $X = M^2 B$.

Which of the following statements is/are **TRUE**?

(A)	$A^T X = 0$ for all $B \in S$
(B)	$M^{2026} B = (A^T A)^{1012} X$
(C)	$X^T M B = 0$ for all $B \in S$
(D)	If $A^T B = 0$ then there exists a scalar λ such that $X = \lambda B$.

Q63.

Let $f:(0,\infty)\to\mathbb{R}$ be a function defined by:

$$f(x) = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \int_1^{x^2} \frac{\sin(\pi\sqrt{t})}{\sqrt{t}}dt$$

Which of the following statement(s) is (are) correct?

(A)	There exists at least one $c \in (2,3)$ such that $f'(c) = 0$.
(B)	$\lvert f(2) - f(1)\rvert = \sin^{-1}\left(\frac{3}{5}\right) + \frac{3}{\pi}$
(C)	$f'(x)$ is strictly increasing in $(0, 1/2)$.
(D)	$\lim_{x\to\infty} f(x)$ does not exist.

Q64.	Let α and β be the roots of the quadratic equation $x^2 - x - 1 = 0$ with $\alpha > \beta$. Define the sequence $\{S_n\}$ for $n \geq 0$ as $S_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$. Which of the following statement(s) is (are) correct?	
	(A)	$\sum_{k=1}^{10} S_k^2 = S_{10} S_{11}$
	(B)	$\sum_{k=1}^{10} \binom{10}{k} 2^k S_k = S_{30}$
	(C)	$\sum_{n=1}^{\infty} \left(\dfrac{nS_n}{10^n} \right) = \dfrac{1010}{(89)^2}$
	(D)	$\lim_{n \to \infty} \left(\dfrac{S_{n+1}}{S_n} \right) = \dfrac{1+\sqrt{5}}{2}$

Q65.	Let $S = \{z \in \mathbb{C} : 3 \leq \lvert 2z - 3(1+i) \rvert \leq 7\}$ be a set of complex numbers. Let $\omega = e^{i\frac{2\pi}{3}}$ be a non-real cube root of unity. Which of the following(s) is (are) true?	
	(A)	$\min_{z \in S} \left\lvert z + \dfrac{5+3i}{2} \right\rvert = \dfrac{3}{2}$
	(B)	There exists $z \in S$ such that $\mathrm{Re}(\omega^2 z) = 0$ and $\lvert z \rvert = \dfrac{3}{\sqrt{2}}$.
	(C)	$\max_{z \in S} \lvert z - i \rvert + \lvert z + i \rvert = 3$
	(D)	There exists a square with side length $\sqrt{2}$ such that all its vertices lie in region S.

Q66.

Let S be the set of all strictly increasing bijective functions f:[0,1]→[0,1] such that f is twice differentiable. Let $f \in S$ satisfy the following functional equation:

$$e^x f(x) + f(e^{-x}) = xe^x + e^{-x} \quad \forall x \in [0,1]$$

Let $a \in (0,1)$ be the unique point in set $\{x \in [0,1] : xe^x = 1\}$.

Let g:[0,1]→[0,1] be the inverse of f, such that $g(f(x))=x$ and $f(g(x))=x$ for all $x \in [0,1]$. Which of the following statement(s) is (are) TRUE?

(A)	$\int_0^1 g(x)dx = \frac{1}{2}$
(B)	$g'(a) = 1$
(C)	$\lim\limits_{x \to a}\left(\frac{fog(x) - a}{x - a}\right) = 1$
(D)	$f''(a) = 1$

Q67.

Let $\phi : (-1, \infty) \to (0, \infty)$ be the solution of differential equation

$\frac{dy}{dx} - 2ye^x = 2e^x\sqrt{y}$ satisfying $\phi(0) = 1$. Then which of the following is/are TRUE?

(A)	There exists some $a \in \mathbb{R}$ such that $\phi(a) > 2026$
(B)	$\lim\limits_{x \to \ln 2} \phi(x) = (2e - 1)^2$
(C)	$\lim\limits_{x \to \ln 2} \phi(x) = \sqrt{2e - 1}$
(D)	ϕ is a strictly increasing function on the interval $(0, \infty)$.

Q68.

Let a, b and c $\in \mathbb{R}$ such that $\lim\limits_{x\to 0}\dfrac{a\tan x + b\tan^{-1}x + c\sin^{-1}x}{x^5} = 3$ Then which of the following (s) is (are) CORRECT?	
(A)	$\lvert a+b\rvert = 40$
(B)	$\lvert b+c\rvert = 30$
(C)	$\lvert a+c\rvert = 10$
(D)	$\lvert a+b+c\rvert = 1$

Q69.

Let P be the parabola defined by the equation $x^2 = 4y$. A sequence of circles $\{C_n : n = 0, 1, 2, \ldots\}$ is constructed such that the circle C_0 is internally tangent to the parabola P at the vertex (0, 0). For each $n \geq 0$ the circle C_{n+1} is externally tangent to the preceding circle C_n. Every circle C_n (for $n \geq 1$) is internally tangent to the parabola P at two distinct points. Let r_n denote the radius and $(0, y_n)$ denote the center of the circle C_n, with $r_0=2$. Which of the following statement(s) is (are) TRUE?	
(A)	$\dfrac{r_{10}^2 - r_9^2}{y_{10} - y_9} = 4$
(B)	$y_{11} - y_{10} = 88$
(C)	$\sum\limits_{k=1}^{10}\left(\dfrac{y_k - y_{k-1}}{k}\right) = 90$
(D)	$\lvert r_{25} - r_{10}\rvert = 60$

Q70.

Let $m, n \in \mathbb{N}\cdot$ Consider Integrals defined by : $B(m,n) = \int_0^1 x^m(1-x)^n\,dx$ $S(m,n) = \int_0^1 \dfrac{x^m(1-x)^n}{(x+1)^{m+n+2}}dx$ Which of the following (s) is (are) TRUE?	
(A)	$S(10,12) = S(12,10)$
(B)	$S(2,2) = \dfrac{1}{240}$
(C)	$S(4,5) = \dfrac{B(4,5)}{32}$
(D)	$S(4,5) = \int_0^{1/2} t^4(1-2t)^5\,dt$

Q71.

The centre A of a circle of radius 1 is chosen uniformly and randomly from the line segment joining (0,0) and (2,0). The centre B of a circle of radius 1 is chosen uniformly and randomly from the line segment joining (0, 1) to (2, 1). If A and B are chosen independently, the probability that the two circles intersect is/are:	
(A)	$\frac{4\sqrt{3}-3}{4}$
(B)	$\frac{4\sqrt{3}}{7}$
(C)	$\frac{\sqrt{3}}{2}$
(D)	$\frac{4\sqrt{3}-3}{5}$

Q72.

For each $t \in \mathbb{R}$ let L_t be line segment joining the points (0, 1) and (t, 0). Let P_t be point of intersection of line segment L_t with parabola $y = x^2$. Define $f : \mathbb{R} \to \mathbb{R}$ as f(t) = y-coordinate of point P_t. Then which of the following (s) is (are) TRUE?	
(A)	f is a bounded function
(B)	f is a continuous function
(C)	$\lim_{t\to\infty} f(t) = 1$
(D)	$f'(0) = 0$ (f is differentiable at t=0)

Q73.

Let T_n denotes number of n digit positive integers divisible by 3 whose digits are from set $\{2,3,7,9\}$ (with or without repetition). Then which is TRUE?	
(A)	$T_6 = 1366$
(B)	$T_{10} - T_9 = 1024$
(C)	$T_7 = 1366$
(D)	$T_5 - T_4 = 256$

Q74.

Let $\mathbb{C}$ denotes the set of complex numbers. Consider the sets S and T defined by : $S = \left\{ \lvert 1+z \rvert + \lvert 1-z+z^2 \rvert : z \in \mathbb{C}, \lvert z \rvert = 1 \right\}$ $T = \left\{ \lvert 1+z^2 \rvert + \lvert 1-z \rvert : z \in \mathbb{C}, \lvert z \rvert = 1 \right\}$ Let min(X) and max(X) denotes the minimum and maximum elements in set X respectively. Which of the following statements is (are) CORRECT?	
(A)	$\min(S) = \sqrt{3}$
(B)	$\max(S) = \frac{13}{8}$
(C)	$\min(T) = \sqrt{3}$
(D)	$\max(T) = 4$

Q75.

Let ω be non-real cube root of unity $(\omega^3 = 1, \omega \neq 1)$ and $\mathbb{Z}$ denotes set of integers. Consider the following sets: $S = \left\{ (a,b) \in \mathbb{Z} \times \mathbb{Z} : \lvert a\omega + b \rvert = 1 \right\}$ $T = \left\{ (a,b,c) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : \lvert a\omega + b\omega^2 + c \rvert = 1 \right\}$ $P = \left\{ a + b\omega + c\omega^2 : a,b,c \in [0,1] \right\}$ Which of the following (s) is (are) TRUE?	
(A)	$\lvert S \rvert = 6$
(B)	Set 'P' represents "Hexagon"
(C)	$\lvert T \rvert = 6$
(D)	Set 'P' represents "Pentagon"

Q76.

Let $f:[0,1]\to\mathbb{R}$ be continuous function and a be a real number such that $\int_0^1 e^{ax}(f(x))^2\,dx = 2\int_0^1 f(x)dx + \frac{e^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4}$ Then which is TRUE?	
(A)	a=2
(B)	a=3
(C)	$\lim\limits_{x\to 0}(f(x)e^{2x}) = 1$
(D)	$\lim\limits_{x\to 1}(f(x)e^{2x}) = 1$

Q77.

Let $f:[0,1]\to R$ be a function defined by: $f(x)=\int_0^x \lvert\sin(5\pi t)-\sin(3\pi t)\rvert\,dt$ Which of the following statement(s) is (are) **CORRECT**?	
(A)	The function f(x) is non-differentiable at exactly 4 points in the interval (0, 1).
(B)	The derivative function f'(x) is non-differentiable at exactly 4 points in the interval (0, 1).
(C)	There exists at least two real number $c\in(0,1)$ such that f''(c) = 0.
(D)	$\int_0^x f(t)dt < \frac{\pi x^3}{3}$ holds true for all $x\in(0,\frac{1}{8})$.

Q78.

Let σ_X denotes the standard deviation of elements of any set X and $\bar{X}$ denotes the mean.

$$A=\left\{\frac{2n+3}{3^n}:n\in\mathbb{N},n\le 20\right\}$$

$$B=\left\{\frac{2n^2+3n-4}{3^n}:n\in\mathbb{N},n\le 20\right\}$$

$$C=\left\{\frac{3n^2+4}{n^2+2}:n\in\mathbb{N},n<20\right\}$$

$$D=\left\{\frac{4n^2+5}{n^2+2}:n\in\mathbb{N},n<20\right\}$$

Then which of the following(s) is(are) CORRECT?

(A)	$\bar{A}=\frac{3}{20}-\left(\frac{23}{20}\right)3^{-20}$
(B)	$4\bar{B}=\frac{13}{9}-\frac{499}{3^{11}}$
(C)	$4\sigma_C=3\sigma_D$
(D)	$3\sigma_C=2\sigma_D$

Q79.

Let C be the parabola given by the equation $y^2 = 4x$, with its focus at S. Tangents are drawn from the point P(-1, 2) to the parabola C, touching it at points A and B. Let r(X) denote the radius of the circle which is tangent to the parabola C at a point X and passes through the focus S. Which of the following statement(s) is(are) TRUE?

(A)	$SA\cdot SB=SP^2$
(B)	$PA\cdot PB=8SP$
(C)	$\text{Area}\,(\Delta PAB)=8\sqrt{2}\,\text{Sq.units}$
(D)	$\left\lvert\frac{r(A)}{r(B)}\right\rvert$ can be $\sqrt{5}+1$

Q80.

Consider the closed region Ω in the first quadrant of the cartesian plane, defined by the set: $\Omega = \left\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, x^2 \leq y \leq \sqrt{2-x^2}\right\}$

Let L_m be a lines passing through origin with slope $m \in (0,\infty)$, defined by equation: y = mx. This line separates Ω into two disjoint sub-regions, defined as:

$G_m = \{(x,y) \in \Omega \mid y \geq mx\}$

$R_m = \{(x,y) \in \Omega \mid y \leq mx\}$

Where G_m and R_m represents green and red region respectively.

If A(S) denote the area of set $S \subset \mathbb{R}^2$. Which of the following (s) is (are) CORRECT?

(A)	There exists at least one $m_0 \in (0,\infty)$ such that $A\left(G_{m_0}\right) = A\left(R_{m_0}\right)$
(B)	There exists at least one $m_1 \in (0,\infty)$ such that $A\left(G_{m_0}\right) = 5\,A\left(R_{m_0}\right)$
(C)	If $A\left(G_m\right) = A\left(R_m\right)$ then $m = \cot\left(\frac{\pi}{8} + \frac{1}{6}\right)$
(D)	If $A\left(G_m\right) = A\left(R_m\right)$, then $A(G_m) = \frac{3\pi+2}{24}$

Q81.

Let P(6,8) be a point outside the circle $S: x^2 + y^2 = r^2$. Tangents PA and PB are drawn to S, where A and B are points of tangency. Let Δ be area of triangle formed by tangents PA, PB and the chord of contact AB. Let R be the region bounded by the tangents PA, PB and the minor arc AB. Which of the following (s) is(are) CORRECT?

(A)	If f(r) represents area of the region R, there exists at least one $r \in (0,10)$ such that $\frac{df}{dr} = 0$.
(B)	Let S_1 be a circle tangent to tangents PA, PB and S and lies in region R. Let $r(S_1)$ denotes radius of such circle S_1. Then $\max\limits_{r\in(0,10)}(r(S_1)) = 5(3-2\sqrt{2})$.
(C)	For r=5, let $C_1, C_2, \ldots, C_n$ be a sequence of circles such that each C_n is tangent to both PA and PB. If C_1 is tangent to minor arc of S externally and C_{n+1} is tangent to C_n externally for all $n \geq 1$, then $\left\|\sum_{n=1}^{\infty} r_n^2\right\| < \frac{25}{\pi}(\sqrt{3} - \frac{\pi}{3})$. (Here r_n is radius of circle C_n)
(D)	$\max\limits_{r\in(0,10)}(\Delta)$ occurs at r=5.

Q82.

Let $f:\mathbb{R}\to\mathbb{R}$ given by f(x) = 2x - sin x for all $x\in\mathbb{R}$ Let $K\in\mathbb{N}$ such that $\lim\limits_{x\to 0}\frac{1}{x}\left(\sum_{i=1}^{K} i^2 f\left(\frac{x}{i}\right)\right)=45.$ Let g(x) = f(f(f(x))) and let $g^{-1}:\mathbb{R}\to\mathbb{R}$ be inverse function of g. Let $H(x)=\int_x^{x^2} f(xt)dt$ for x > 0 and $\phi(x)=\sin\left(\frac{\pi}{x}\right),\quad x\neq 0$ Then which of the following (s) is (are) TRUE?	
(A)	$(g^{-1})'(0)=1$
(B)	$H'(1)=2-\sin 1$
(C)	There exists infinitely many $c\in\left(0,\frac{1}{K}\right)$ such that $\phi'(c)=0$
(D)	$\lim\limits_{x\to 0^+}\frac{H(x)}{x^3}=\frac{-1}{2}$

Q83.

Let F be the family of ellipses E(a, b) defined by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ Every member of F is tangent to the fixed line $L: y=x+\sqrt{13}.$ Let e be the eccentricity, A be the area and $P(x_0,\ y_0)$ be the point of intersection of tangents drawn at the end points of latus rectum for any member $E\in F.$ Which of the following (s) is (are) CORRECT?	
(A)	There are exactly two such ellipse in F if $(a,b)\in\mathbb{N}\times\mathbb{N}.$
(B)	If T(h, k) denotes the point outside ellipse $E\in F$ such that tangents drawn to E is mutually perpendicular. Then $\max\|h+k\|=\sqrt{26}$
(C)	The maximum possible area A of ellipse $E\in F$ is $\frac{13\pi}{2}.$
(D)	$\lim\limits_{b\to 0^+}(x_0)=\sqrt{13}$ for $E\in F$ satisfying a > b.

Q84.

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by: $f(x)=\begin{cases}(1-x)^2\left((1-x)^4\sin(x^2)+e^{\sin(\pi x^2)}-1\right), & x\in(0,1)\\ 0, & \text{otherwise}\end{cases}$ and f' be its derivative. Let $S=\{c\in\mathbb{R}: f'(x)\le cf(x) \text{ for all } x\in\mathbb{R}\}$. Then which of the following (s) is(are) TRUE?	
(A)	$S=\phi$
(B)	$S\ne\phi$ and $S\subset(1,\infty)$
(C)	$(2,\infty)\subset S$
(D)	$S\cap(0,1)\ne\phi$

Q85.

Let $\mathbb{R}_0^+=\{x\in\mathbb{R}: x>0\}$. Define the following subsets in the xy-plane: $S=\{(x,y)\in\mathbb{R}_0^+\times\mathbb{R}_0^+ : y\le\sin(\pi x), x\le 1\}$ $R_n=\bigcup_{k=1}^{n}\left\{(x,y)\in\mathbb{R}^2:\frac{k-1}{n}\le x\le\frac{k}{n}, 0\le y\le\sin(\frac{k\pi}{n})\right\}$ For each $n\in\mathbb{N}, R_n$ is union of n rectangles. $Q=\left\{(x,y)\in\mathbb{R}_0^+\times\mathbb{R}_0^+ : y^4+4xy^3+6x^2y^2+4x^3y+x^4=x^2\right\}$ Let $\lVert X\rVert$ denotes area enclosed by the set X. Which of the following(s) is(are) CORRECT?	
(A)	$\lVert S\rVert=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{\infty}\sin(\frac{k\pi}{n})$
(B)	The sequence of areas $\lVert R_n\rVert$ converges to L such that $L>\lVert S\rVert$.
(C)	The set Q represents a portion of parabola whose area $\lVert Q\rVert$ is $\frac{1}{6}$ sq. units.
(D)	$\lim_{n\to\infty}(\lVert R_n\rVert-\lVert S\rVert)=0$

Q86.

Let E_1, E_2 and E_3 be three independent events (representing three students passing an exam) with probabilities $P(E_1) = p_1$, $P(E_2) = p_2$ and $P(E_3) = p_3$.

Let X be a discrete random variable denoting the total number of events among $\{E_1, E_2, E_3\}$ that occur. The probability distribution of X is given below:

Which of the following(s) is (are) CORRECT?

x	0	1	2	3
P(X=x)	$\frac{2}{5}$	$\frac{13}{30}$	$\frac{3}{20}$	$\frac{1}{60}$

(A)	The values of p_1, p_2, p_3 are $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ in any order.
(B)	The probability that at least two students pass the exam is $\frac{1}{6}$.
(C)	The events E_1, E_2 and E_3 are equally likely.
(D)	The expected value E(X) of the number of students who pass is $\frac{47}{60}$.

Q87.

Consider the plane P defined by the equation $x + y - z = 1$.
Let A(1, 0, 1) and B(1, r, 1) be two fixed points in space where $r \in \mathbb{R} \setminus \{1\}$.
A variable point X(x, y, z) moves on the given plane P such that the line segments AX and BX are inclined at equal angle to the plane P. Let L(r) denote the resulting locus of point X. Which of the following (s) is (are) CORRECT?

(A)	If r = 2, the locus L(r) is a straight line.
(B)	If $r = \sqrt{3} + 1$, the locus L(r) is an ellipse.
(C)	If r = 4, the locus L(r) is a circle.
(D)	If r = 0, the locus L(r) is a straight line.

Q88.

Let $E: \frac{x^2}{2} + \frac{y^2}{1} = 1$ be an ellipse with left and right foci F_1 and F_2 respectively. Let $P(x_0, y_0)$ be a variable point on E such that $x_0 > 0$, $y_0 > 0$. The lines PF_1 and PF_2 intersects the ellipse E again at points $Q_1(x_1, y_1)$ and $Q_2(x_2, y_2)$ respectively. Let r_1 denotes inradius of $\triangle PF_1Q_2$ and r_2 denotes the inradius of $\triangle PF_2Q_1$. Which of the following statement(s) is (are) CORRECT?

(A)	The perimeter of $\triangle PF_1Q_2$ is $4\sqrt{2}$
(B)	$r_1 - r_2 = \frac{y_1 - y_2}{2\sqrt{2}}$
(C)	$\max(r_1 - r_2) = \frac{\sqrt{2} - 1}{2}$
(D)	$\max(r_1 - r_2) = \frac{1}{3}$

Q89.

Let C be the curve defined by $y=x^2$ for $x \in [0,1]$.

For any real number $b \in [0,1]$, let A(b) denotes the area of the region bounded by curve C, the lines x=0, x=1 and the horizontal line y=b.

Let b_0 be the value of b for which A(b) is minimized.

Let Ω be the corresponding geometrical region for this minimal case.

Which of the following statement(s) is(are) TRUE?

(A)	$b_0 = A(b_0)$
(B)	The locus of a point P in xy-plane that is equidistant from $y=b_0$ and the point $F(0, 2b_0)$ is a parabola with vertex $(0,\frac{3}{8})$.
(C)	There exists some $c \in (0,1)$ such that $A(c)=\frac{20}{26}$.
(D)	$L = \Omega \cap \{(x,y): x \le \sqrt{b_0}\}$ $R = \Omega \cap \{(x,y): x \ge \sqrt{b_0}\}$ A(X) represents area bounded by points in set X. $\frac{A(L)}{A(R)} = \frac{1}{3}$

Q90.

Let $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ where $\cos^{-1} x \in [0,\pi]$.

Define $s_n = \frac{(\sqrt{3})^n \sin(n\theta)}{\sqrt{2}}, n \in \mathbb{N}$ and $c_n = (\sqrt{3})^n \cos(n\theta)$

Then which of the following(s) is(are) TRUE?

(A)	$s_{20} \in \mathbb{N}$ and $c_{26} \in \mathbb{N}$
(B)	$\lvert s_4 + c_4 \rvert = 5$
(C)	$\sum_{n=0}^{\infty} \frac{s_n}{3^n} = \frac{1}{2}$
(D)	$\frac{s_{12} + 3s_{10}}{s_{11}} = 2$

Q91.

Let E_1 and E_2 be two elliptical regions given by: $E_1=\left\{(x,y)\in\mathbb{R}^2:\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}$ $E_2=\left\{(x,y)\in\mathbb{R}^2:\frac{x^2}{b^2}+\frac{y^2}{a^2}\le 1\right\}$ Given $a>b>0$, the union $E_1\cup E_2$ is partitioned into five non-overlapping regions R_0,R_1,R_2,R_3 and R_4 such that: $R_0=S_1\cap S_2$ $R_1=\{(x,y)\in S_1\setminus S_2:x<0\}$ $R_2=\{(x,y)\in S_1\setminus S_2:x>0\}$ $R_3=\{(x,y)\in S_2\setminus S_1:y<0\}$ $R_4=\{(x,y)\in S_2\setminus S_1:y>0\}$ If area bounded by region bounded by set of points in set X be denoted by A(X). If $A(R_0)=A(R_1)=A(R_2)=A(R_3)=A(R_4)$. Then which of the following statement(s) is (are) CORRECT?	
(A)	$\frac{a}{b}=2+\sqrt{3}$
(B)	The area of square formed by four points of intersection of E_1 and E_2 is $\frac{2a^2b^2}{a^2+b^2}$.
(C)	$A(R_0)=4ab\tan^{-1}\left(\frac{b}{a}\right)$
(D)	$A(E_1\cup E_2)=\frac{5}{3}\pi ab$

Q92.

Let L(a, b) denotes locus of point p such that the tangents drawn from it to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the coordinate axes at concyclic points. Then	
(A)	L(2025, 2025) is pair of straight lines
(B)	L(2026, 2025) is a hyperbola
(C)	L(2025, 2024) is a hyperbola
(D)	L(2025, 2025) is a circle

NUMERICAL VALUE

Q.93

Let M be the set of all 3×3 matrices $A = [a_{ij}]$ with entries in $\{0, 1\}$ such that:

$\sum_{k=1}^{3} a_{ik} = 1 \quad \forall\, i \in \{1,2,3\}$ and $\sum_{k=1}^{3} a_{kj} = 1 \quad \forall j \in \{1,2,3\}$.

Consider a function $\psi : M \to \mathbb{R}$ defined as $\psi(A) = \det(A) \cdot \text{trace}(A^{2026})$

Then the value of $\left| \sum_{A \in M} \psi(A) \right|$ is/are_______

Q.94

Let $x_n = \dfrac{1}{n^2 + n + 1}$ for all $n \in \mathbb{N}$. Consider function f(x) defined by:

$$f(x) = \frac{3}{2}\cos^{-1}\sqrt{\frac{1}{1+x^2}} + \frac{1}{4}\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \tan^{-1} x$$

Find the value of $\dfrac{8}{\pi}\left|\sum_{n=1}^{\infty} f(x_n)\right|$

Q.95

Let L_1 and L_2 denote the lines:

$\vec{r}_1 = -9\hat{i} + 7\hat{j} + 6\hat{k} + \lambda(\hat{i} - \hat{j})$

$\vec{r}_2 = 7\hat{i} + 3\hat{j} + 4\hat{k} + \mu(\hat{i} + 3\hat{k})$

Let S be the Sphere whose diameter is the line segment connecting the points of shortest distance between L_1 and L_2. Let Π be the plane tangent to the Sphere S at the point where S touches the line L_1. If d be the perpendicular distance from origin (0, 0, 0) to the plane Π then the value of $19d^2$ is ____

Q.96

Let S be set of all non-zero 3×3 matrices $A=[a_{ij}]$ with integer entries $(a_{ij}\in\mathbb{Z})$ such that $X^TAX=0$ is true for all Column vector $X\in\mathbb{R}^3$. If $A^5=A$ for all $A\in S$ and I be Identity matrix of order 3×3. $\left|\sum_{A\in S}\det(A+I)\right|$

Q.97

Let S be the set defined by

$$S=\left\{(48+81i)z+\frac{96+144i}{z}:z=a+ib \text{ where } (a,b)\in\mathbb{R}\times\mathbb{R}\right\}.$$

If |z|=4 then largest possible value of Real part of elements in Set S is/are ________

Q.98

Consider the Region S in the cartesian plane defined by

$$S=\left\{(x,y)\in\mathbb{R}\times\mathbb{R}:0\le x\le1, 0\le y\le1 \text{ and } \frac{1}{4}\le xy\le\frac{1}{2}\right\}.$$

If area of region S is t. Then |32t| is/are

Q.99

Let E be the ellipse defined by equation $\frac{x^2}{3}+y^2=1$. Let S and S' be foci of the ellipse E (S lies on positive x-axis). Consider a point P on ellipse in first quadrant such that normal at P passes through the lower end of the minor-axis. Let M, N be feet of perpendiculars dropped from S and S' respectively ONTO tangent at P. Then $\frac{(SN)^2+(S'N)^2}{(SM)(S'N)}$ is______

Q.100 For a positive integer n, let a_n, b_n, c_n, d_n be the real numbers such that $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$. If $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$, then $|L^2 - \frac{1}{L}|$ is______

Q.101 Let S be the set of all continuous functions $f:[0,1]\to\mathbb{R}$ that satisfy $\int_0^1 f(x)(1-f(x))dx = \frac{1}{4}$. Then |S| is ________

(Note: |X| denotes number of elements in set X).

Q.102 Let $E = \{(x,y)\in\mathbb{Z}\times\mathbb{Z} : \frac{x^2}{8}+\frac{y^2}{20}<1\}$

$H = \{(x,y)\in\mathbb{Z}\times\mathbb{Z} : y^2 - x^2 \geq 1\}$

$C = \{(x,y)\in\mathbb{Z}\times\mathbb{Z} : x^2 + (y-1)^2 \leq 4\}$

Consider $X = E\cap H\cap C$.

Let S be the set of all possible triplets of distinct points chosen from X.

Let E be the event that the area of triangle ΔPQR formed by triplet $\{P, Q, R\}$ is a positive integer. $E = \{\{P,Q,R\}\in S : \text{Area}(\Delta PQR)\in\mathbb{Z}^+\}$.

Then Probability of Event E is ________

Q.103 Let T be the parabola described by equation $y^2 = 4x$. Let S be the set of all points P(h, k) in plane from which three distinct normals can be drawn to T. Let the slopes of these normals be denoted by the set $M = \{m_1, m_2, m_3\}$. A specific constraint is imposed on the set M such that product of two slopes is a constant α (that is $m_i m_j = \alpha$ for some $i \neq j$). If this constraint forces the locus of point P to lie entirely on the curve T, then the value of α is ________

Q.104

The value $2025\left|\dfrac{\int_0^1 (1-x^{45})^{44}dx}{\int_0^1 (1-x^{45})^{45}dx}\right|$

Q.105

For any integer k, let α_k be the complex 2026^{th} root of unity defined as:

$\alpha_k = e^{\frac{ik\pi}{1013}}$.

Then $\frac{1}{4}\left(\sum_{k=1}^{2026}\left(|\alpha_k - \alpha_{k+137}|^2 + |\alpha_k + \alpha_{k+137}|^2\right)\right)$ is_____

Q.106

The largest positive number a such that $\int_0^5 f(x)dx + \int_0^3 f^{-1}(x)dx \geqslant a$ for a every strictly increasing surjective continuous function $f:[0,\infty)\to[0,\infty)$ is :

Q.107

$S = \{x \in [0,2\pi] | \sin(14x) = \cos(68x)\}$.

Then number of elements in set 'S' is/are

Q108.

Consider a family of lines $\{L_n\}$ where $n \in \{1,2,3,\cdots,10\}$.

Each line L_n is defined by vector equation $L_n : \vec{r} = n\hat{i} + \lambda(\hat{j} + \sqrt{n}\hat{k}), \lambda \in \mathbb{R}$.

Let d_n denotes shortest distance between L_n and L_{n+1}.

Then $\left|\sum_{n=1}^{10}\frac{1}{d_n^2}\right|$ is/are

Q109.

Let $f:\{1,2,\ldots,10\}\to\mathbb{R}$ be a function that satisfies

$\left(\sum_{i=1}^{10}\frac{|f(i)|}{2^i}\right)^2 = \left(\sum_{i=1}^{10}|f(i)|^2\right)\left(\sum_{i=1}^{10}\frac{1}{4^i}\right)$.

If f(10) = 1 and $f(i)\in\mathbb{Z}$ for all $i \in \{1,2,\ldots,10\}$,

then the total number of such functions f is/are

Q110. Let r_1, r_2, r_3 be the three roots of the polynomial $p(x) = x^3 - 6x^2 + \alpha x - \beta$ where $\alpha, \beta \in \mathbb{Z}$. If roots r_1, r_2, r_3 are distinct integers in Arithmetic progression and β is largest possible negative integer. Then $\left|\sum_{i=1}^{3} (r_i)^4\right|$ is/are

Q111. Let $S = \{z \in \mathbb{C} : z^9 = 1\}$ be set of the 9^{th} roots of unity. Let F be family of all subsets $T \subseteq S$ such that |T| = 3. Then $\left|\sum_{T\in F}\left(\sum_{z\in T} z^2\right)\right|$ is/are (Note: |X| denotes number of elements in set X)

Q112. Denote

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad B = \begin{bmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix}$$

If $aei + bgf + cdh - afh - bdi - ceg = (2)^{\frac{1}{5}}$

Let $\text{adj}(\text{adj}(\text{adj}(B))) - \lambda A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The value of λ is__________________

Q113. $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 2 \le |3x + y + 7| + |x + 5y + 9| \le 8\}$.
The area enclosed by points in set S is A.
Then the value of |7A| is ______

Q114. Let S be the set of all ordered 5-tuples defined as: $S = \{(a, b, c, d, e) : a, b, c, d, e \in \{-1, 0, 1\}\}$

Let $w = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$ be a complex number where $i = \sqrt{-1}$. Define

$V = \{\alpha \in \mathbb{C} : \alpha = a + bw + cw^2 + dw^3 + ew^4, \text{ where } (a, b, c, d, e) \in S\}$.

Then the number of elements in set T given by
$T = \{(a, b, c, d, e) \in S : |\alpha| = 1\}$ is/are______

Q115. $9\left|\int_0^1\left(\sqrt{x+x^3}+\sqrt{x+\sqrt[3]{x}}\right)dx\right|^2$ is

Q116. Let $\mathbf{f(t)=\prod_{n=2}^{\infty}\left(1+\frac{t}{n^2+n}\right)^{\frac{1}{t}}}$.

If $|\lim_{t\to 0} f(t)|=L$. Then $16(\log_e L)^2$ equals

Q117. The value of

$\frac{16}{\pi}\left|\int_0^1 \sin^{-1}\sqrt{\frac{2}{\pi}\sin^{-1}(\sqrt{x})}\,dx\right|$ is :

Q118. Let $f_n:\mathbb{R}\to\mathbb{R}$ be a sequence of functions defined by

$f_n(x)=\prod_{k=1}^{n}\cos(kx)$ where $n\in\mathbb{N}$.

The smallest number 'n' such that $|f_n''(0)+f_n'''(0)|>2026$ is

Q119. Let T(n) denotes the number of n-digit numbers (using 0, 1, 2, 3,…, 9) such that no three consecutive digits are same.

Then $\left|\frac{T(12)}{T(10)+T(11)}\right|$ is

Q120. Let $S_i(j)=\frac{2026}{4051}\frac{\binom{2025}{i}\binom{2025}{j}}{\binom{4050}{i+j}}$

The value of $\sum_{j=0}^{2025}S_i(j)$ is:

Q121.

The value of $\left|\frac{1}{1-\cos\left(\frac{\pi}{9}\right)}+\frac{1}{1-\cos\left(\frac{5\pi}{9}\right)}+\frac{1}{1-\cos\left(\frac{7\pi}{9}\right)}\right|$ is

Q122.

If

$$\left|\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{n-k}{n^2}\cos\left(\frac{4k}{n}\right)\right)\right|=\frac{\sin^2 t}{4t}.$$

Then t is

Q123.

Let

$T_n=(2+\sqrt{3})^n$ for $n\in\mathbb{N}$. Denote $f_n=T_n-[T_n]$.

Then $\left|\frac{(f_5)^2}{1-f_5}\right|$ is

(Note: $[T_n]$ is Greatest integer function)

Q124.

Let M = {1, 2} be a finite set. Let P(M) denotes power set of M, and F = P(P(M)) be the set of all possible collections of subsets of M. A collection $S\in F$ is such that it satisfies the property $\bigcup_{A\in S} A = M$. The number of such unique collections S is

Q125.

The value of

$$\int_0^{2026}\min\left(\sqrt{\{x\}-\{x\}^2},\left|\frac{\sqrt{3}\sin^{-1}(\cos\pi x)}{\pi}\right|\right)dx$$

(Note: {x} represents fractional part of x)

Q126.

Let f be a continuous function satisfying:

$f(x^7+6x^5+3x^3+1)=9x+5$ for all real x.

Then $\left|\int_{-9}^{11}f(x)dx\right|$ is

Q127. $\left|\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt[4]{2}+\sqrt{1+\tan x}}dx\right| = \frac{\pi}{a\sqrt[4]{2}}, a \in \mathbb{N}.$

Then a is

Q128. Let P be the parabola $(y - 20)^2 = 4a(x - 26)$ with vertex A and let B and C be two other distinct points on P. The focus F of P is the centroid of ΔABC. If FB = 3, Then the area of triangle Δ ABC be t.

Then $\left|\frac{625}{2916}t^2\right|$ is

Q129. The value of

$29\cos\left(\sum_{k=1}^{40}\cos^{-1}\left(\frac{k^2+k+1}{\sqrt{k^4+2k^3+3k^2+2k+2}}\right)\right)$ is equal to:

Q130 . Consider the Lines L_i in $\mathbb{R}^3$ (3-D plane) for i = 1, 2, 3 as:

L_1: x - 10 = 11 - y = z - 12

L_2: $x - 11 = 10 - y = \frac{13-z}{3}$

$L_3 : \frac{x-10}{1} = \frac{y-11}{1} = \frac{z-12}{0}$

Let $H(\alpha,\beta,\gamma)$ denote the orthocenter of triangle formed by points $L_1 \cap L_2, L_2 \cap L_3$ and $L_3 \cap L_1$. Then $|\alpha+\beta+\gamma|$ equals

Q131. $A = \{x \in \mathbb{C} | \ x^3 + x + 1 = 0\}$

$B = \{x^2 \in \mathbb{C} | \ x \in A\} = \{x_1, x_2, x_3\}$

If $x_1, x_2, x_3 \in B$, are roots of a cubic polynomial P(x) satisfying P(0) = -1,

then |P(9)| is

Q132. Let A = {1, 2, 3}. Let F denotes family of ellipses given by:

$$F = \left\{ E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } (a, b) \in A \times A \text{ and } a \neq b \right\}$$

Let P be a fixed parabola defined by $y = 4x^2$.

For any Ellipse $E \in F$, Let S_E be the set of points on the parabola P such that the tangent to P at that point is also a normal to the ellipse E.

Then find the value of: $\left| \sum_{E \in F} n(S_E) \right|$ (Here: n(X) denotes number of elements in set X)

Q133. A triangle is inscribed in the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ such that its centroid lies at the center of the ellipse (0, 0). If the equation of the locus of the circumcenter of this triangle is $144x^2 + 64y^2 = \lambda$. Then λ is:

Q134. Let $g : (0, \infty) \to \mathbb{R}$ be a differentiable function defined by :

$$g(x) = \int_{\pi}^{x^2} \frac{\sin(\pi\sqrt{t})}{t} dt$$

Let $x_1 < x_2 < x_3$ be the first three positive roots of the equation g'(x) = 0.

Consider the A_g and B_g defined over the closed interval $[x_1, x_3]$:

$$A_g = \{x \in [x_1, x_3] | \; g''(x) = 0\}$$

$$B_g = \{x \in [x_1, x_3] | \; g'''(x) = 0\}$$

The value of $n(A_g) + n(B_g)$ where n(S) denotes the number of elements in set S.

Q135. Let $S=\left\{(x,y): x^2 \le y \le \sqrt{x}, x \ge 0\right\}$

Let Q be the square of maximum area contained entirely within S.

If the area of Q is A and $A=m+n\sqrt{5}, m,n \in \mathbb{N}$. Then |m - n| is

Q136. Let the equation of three lines be given by:

$L_1 : \frac{x}{1}=\frac{4-y}{2}=\frac{6-z}{3}$

$L_2 : \frac{x-3}{2}=\frac{y-3}{1}=\frac{z-3}{0}$

$L_3 : \frac{4-x}{3}=\frac{y+4}{6}=\frac{8-z}{5}$

Let P be the point of intersection of L_1, L_2 and L_3.

Three points A_1, A_2 and A_3 are located on L_1, L_2 and L_3 respectively such that

$PA_1 = PA_2 = PA_3 = 5$ units.

Consider a variable line L passing through P with direction unit vector $\hat{u}$.

Let the points A'_1, A'_2, A'_3 be the reflections of A, B, C respectively in line L.

Three planes P_1, P_2, P_3 are constructed such that plane P_k passes through A'_k and is perpendicular to line L_k for k = 1, 2, 3.

The locus of point of intersection of the planes P_1, P_2 and P_3 as the line varies is a planar region bounded by a polygon. If the area of the polygon is $\sqrt{3}t$ square units, the value of t is:

Q137. Consider a triangle ABC in the xy-plane $(\mathbb{R}^2)$. The side BC is fixed on the x-axis with vertices $B\left(-\frac{1}{2},0\right)$ and $C\left(\frac{1}{2},0\right)$. The vertex A(x, y) moves in the upper half plane (y > 0) such that the orthocenter H of the triangle always lies on the line $y=\frac{1}{4}$. Let S denotes the area of triangle ABC. If the set of all possible values of S for which triangle ABC is acute-angled is the interval (m, M], then the value of $\frac{1}{m}+\frac{1}{M}$ is :

Q138. Let $S_k = \left\{x \in (0, k\pi) : (\log_{\cos x} \sin^2 x)(\log_{\cos^2 x} \sin x) = 4\right\}$.
If the number of elements in set S_k is 5 for $k \in \mathbb{N}$.
Then the sum of all possible distinct values of k is :

Q139. Let $\alpha = \frac{\pi}{3}$. We define sequence s_n for n $\in \mathbb{N} \cup \{0\}$ as:

$$s_n = \frac{\sin(n\alpha)}{\sin\alpha}$$

Consider two 3 ×3 matrices A and B as:

$$A = \begin{bmatrix} s_2 & 3s_3 & s_6 \\ 3s_1 & 3s_5 & s_1 \\ 2s_0 & s_2 & 4s_3 \end{bmatrix}, \quad B = \begin{bmatrix} 2s_2 & 3s_3 & 4s_6 \\ 4s_1 & 4s_5 & 2s_1 \\ 3s_0 & 2s_2 & 4s_3 \end{bmatrix}$$

Let S(M) denotes the sum of all elements in a matrix M.

Then $\left|S\left((AB)^3 - 3(A^{-1}B)^2 - 8(A^{-1}B^{-1})\right)\right|$ is:

Q140. Let

$$S = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 9\pi^2\right\}$$

$$T = \left\{(x, y) \in \mathbb{R}^2 : \sin(x + y) \ge 0\right\}$$

Let A(X) denotes area bounded by collection of points in set X.

If $|A(S \cap T)| = \frac{\lambda\pi^3}{2}$. Then λ is:

Q141. Let z = x + iy be a complex number where x and y are integers.

Let S be the set defined by: $S = \left\{z : |z^2 + 1| = |z + 1|^2\right\}$

If n(S) denotes the number of elements in set S, find the value of: $\left|n(S) + \sum_{z \in S} |z|^2\right|$

Q142. Let S be the set of all possible arithmetic progression (A) having 9 terms that contains the number 6 and 10 as terms. For each such progression A, let f(A) denotes the last term of progression.

Then $\left|\sum_{A\in S} f(A)\right|$ is

Q143. Let S be the set of all 3 × 3 invertible matrix with entries 0 and 1. For each M $\in$ S let $n_0(M)$ and $n_1(M)$ denotes number of 0's and 1's in M.

$\min_{M\in S} |n_1(M) - n_0(M)| + \max_{M\in S} |n_1(M) - n_0(M)|$ is

Q144. Let $f(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$. For any positive integer n define the integral:

$$k_n = \int_0^\infty \frac{1}{(e^x + 2 + e^{-x})^n}dx$$

If $k_{13} = \frac{f(\alpha,\beta)}{\lambda}$, where $\alpha,\beta,\lambda \in \mathbb{N}$ and $\alpha = \beta$, then $|\alpha + \lambda|$ is

Q145. Let S be the set of monic polynomials in x of degree 6 all of whose roots are members of the set {-1, 0, 1}. Let P be the sum of the polynomials in S. Then the coefficient of x^4 in P(x) is:

Q146. Let f(r) be a real valued function defined for r $\in \mathbb{N}, r \geq 3$ as:

$$f(r) = \tan^{-1}\left(\frac{18r}{9r^4 - 15r^2 + 10}\right)$$

where $\tan^{-1}$ denotes inverse tangent function defined as: $\tan^{-1} : \mathbb{R}^+ \to \left(0, \frac{\pi}{2}\right)$

Then $17\tan\left(\sum_{r=3}^{\infty} f(r)\right)$ is

Q147. Let E be the ellipse given by the equation:

$$\frac{x^2}{676}+\frac{y^2}{400}=1$$

Let F_1 and F_2 denote the foci of ellipse E. Consider a variable point P moving on E. Let r(P) denotes the inradius of the triangle PF_1F_2.

If the maximum value of r(P) attain the form $\frac{\lambda e}{1+e}$,

where e is eccentricity of ellipse E, then the value of λ is:

Q148. Consider the following curve C and line L as follows:

$$C:(x^2+y^2-1)=\left(\sqrt{\frac{5}{3}}x+\frac{1}{\sqrt{3}}y-1\right)^2$$

$$L:\frac{x}{\sqrt{5}}+\frac{y}{\sqrt{3}}=1$$

Let line L and curve C intersects at two distinct points P and Q.

Let H(XYZ): orthocenter of triangle formed by ponts X, Y, Z.

I(XYZ): Incenter of triangle formed by points X, Y, Z.

If $H(PQR)=(\alpha,\beta)$ and I(PQR) = (a, b) where point $R=\left(\sqrt{\frac{5}{3}},\frac{1}{\sqrt{3}}\right)$.

Then $\left|\alpha^2+\beta^2-\left(\frac{a}{b}\right)^2\right|$ is

Q149. Let $S=\left\{(x,y)\in\mathbb{R}^2:x^2+y^2-6x-8y+21\le 0\right\}$

$$T=\left\{\frac{12x-5y}{7}:(x,y)\in S\right\}$$

$$P=\left\{\frac{\sqrt{3}y+|x-3|}{|x-3|}:(x,y)\in S\right\}$$ Then $|\max(T)+\min(P)|=$

Q150. $S=\left\{\theta\in[0,2\pi]:8^{2\sin^2\theta}+8^{2\cos^2\theta}=16\right\}$

Then $\left|\sum_{\theta\in S}\left(\sec\left(\frac{\pi}{4}+2\theta\right)\operatorname{cosec}\left(\frac{\pi}{4}+2\theta\right)\right)\right|$

Q151. Let E_1 be the image of ellipse $E: \frac{x^2}{9} + \frac{y^2}{4} = 1$ in line $L: x + y + \sqrt{5} = 0$. A line $y + 2\sqrt{5} = 0$ cuts E_1 at points P and Q. Tangents drawn at P, Q to ellipse E_1 meets in R. If area of (ΔPQR) is $= \Delta$. Then $\frac{256}{9\Delta^2}$ is

Q152. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by:

$f(x) = x^3 - 9x^2 + 24x - 10$

If $f(\alpha) = 8 - \sqrt{5}$ and $f(\beta) = 8 + \sqrt{5}$, then $|\alpha + \beta|$ is

Q153. Let $f: \mathbb{R} \to \mathbb{R}$ be defined as:

$$f(x) = \left(\sin\left(\frac{kx}{10}\right)\right)^4 + \left(\cos\left(\frac{kx}{10}\right)\right)^4, \quad k \in \mathbb{N}.$$

If for any $a \in \mathbb{R}$, $\{f(x) | a < x < a+1\} = \{f(x) | x \in \mathbb{R}\}$,

then the minimum value of k

Q154. Let $f: A \to A$ be a bijective function on set $A = \{1, 2, 3, 4, 5, 6, 7\}$. Define $S = \sum_{n=1}^{7} \max\{f(1), f(2), f(3), \ldots, f(n)\}$. Let N be the number of functions f such that S = 47.[Here max(a, b, c, …..) is maximum among a, b, c, …..]. Then N is (are):

Q155. Let $f: \mathbb{N} \to \mathbb{N}$ and $f(x_2) > f(x_1) \forall x_1, x_2 \in \mathbb{N}$ and $x_2 > x_1$. If $f(f(n)) = 3n \forall n \in \mathbb{N}$. Then f(2) is

Q156.

Let $P(x)=\sum_{r=0}^{n} a_r x^r$ be a polynomial of degree n, where $a_r=\binom{n}{r}$.

Let $F(f(x))=\int_0^1 f(x)dx$. Define α and β as follows:

$$\alpha=F\left(\sum_{r=0}^{n} a_r^2 x^r\right)$$
$$\beta=F\left(\sum_{r=0}^{n-1} a_r a_{r+1} x^{r+1}\right)$$

If $5\alpha=6\beta$, then the value of n is:

Q157.

Let P be the set of all geometric progressions of the form

$G=\{a_1,a_2,a_3,\ldots,a_n\}$, $n\in\mathbb{N}$ such that common ratio $r\in\mathbb{Z}^+\setminus\{1\}$ and $a_n\in\mathbb{Z}^+$ for all n.

Let $S\subset P$ be the collection of sequence satisfying $\{a_1,a_2,a_3\}\cap\{10\}\neq\phi$

Let function $f: S\to\mathbb{Z}^+$ be given by

$f(G) = a_1 + a_2 + a_3$

If $M=\max\{f(G):G\in S, r\le 10\}$ and $m=\min\{f(G):G\in S\}$

Then $\left|\frac{M+m}{5}\right|$ is :

Q158.

Let w = x + iy be a complex number where x, y$\in\mathbb{Z}$.

$$S_1=\{w\in\mathbb{C}:|w-(3+4i)|=1\}$$
$$S_2=\{w\in\mathbb{C}:|w-3|+|w+3|=10\}$$

If $d=\min|w_1-w_2|$ for $w_1\in S_1$ and $w_2\in S_2$.

Then the value of d^2 is :

(**Note:** $\mathbb{Z}$ denotes set of integers and $\mathbb{C}$ denotes set of complex numbers)

Q159.

The value of $\frac{24}{\pi}\int_0^{\sqrt{2}}\frac{(2-x^2)}{(2+x^2)\sqrt{4+x^4}}dx$ is equal to:

Q160. Consider a system of homogeneous linear equations represented by

$$\begin{bmatrix} 13 & b & c \\ a & 23 & c \\ a & b & 42 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

It is given that $a \neq 13, b \neq 23$ and $c \neq 42$; and the system admits a solution (x, y, z) such that $x \neq 0$.

Then the value $\left|\frac{13}{a-13} + \frac{23}{b-23} + \frac{42}{c-42}\right|$ is:

Q161. Consider a non-degenerate tetrahedron T in $\mathbb{R}^3$. The six edges of the Tetrahedron T is given as:

$$L_1 : \vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(2\hat{i} - 3\hat{j} + 3\hat{k})$$
$$L_2 : \vec{r} = (3\hat{i} - \hat{j} + 2\hat{k}) + \lambda(-5\hat{i} + 4\hat{j} + 2\hat{k})$$
$$L_3 : \vec{r} = (-2\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - \hat{j} - 5\hat{k})$$
$$L_4 : \vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(3\hat{i} + 3\hat{j} - \hat{k})$$
$$L_5 : \vec{r} = (3\hat{i} - \hat{j} + 2\hat{k}) + \lambda(\hat{i} + 6\hat{j} - 4\hat{k})$$
$$L_6 : \vec{r} = (-2\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(6\hat{i} + 2\hat{j} - 6\hat{k})$$

For $\lambda \in [0,1]$ where λ is a parameter along each edge segment.

Let N be the total number of distinct planes which is equidistant from all vertices of Tetrahedron T. Then N is ______.

Q162. Let S be the set of all 2 × 2 matrices $A = \begin{bmatrix} a & b \\ 1 & 1 \end{bmatrix}$ where a, b $\in \mathbb{Z}$ and $|b| \leq 50$.

Let $G \subset S$ be the subset of matrices such that for every $A \in G$, the inverse A^{-1} exists and $A^{-1} = [a_{ij}]_{2\times2}$ satisfies $a_{ij} \in \mathbb{Z}$ for all $i, j \in \{1,2\}$.

The number of elements in set G is ______.

Q163. Let $f(x) = x^6 - 2x^5 + x^3 + x^2 - x - 1$ and $g(x) = x^4 - x^3 - x^2 - 1$ be two polynomials. Let a, b, c, d be roots of g(x) = 0. Then the value of |f(a) + f(b) + f(c) + f(d)| is

Q164. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by: $f(x) = \frac{x^4}{12} + \cos x + 1$

Let $S_0, S_1, S_2, \ldots, S_n$ be the sets of real roots for f(x) and its successive derivatives, such that:

$S_0 = \{x \in \mathbb{R} : f(x) = 0\}$
$S_1 = \{x \in \mathbb{R} : f'(x) = 0\}$ and so on …
$S_2 = \{x \in \mathbb{R} : f''(x) = 0\}$

The smallest integer k such that the set S_k contains infinitely many elements.

Q165. Let $\hat{a}$ and $\hat{b}$ be two unit vectors in $\mathbb{R}^3$ such that $|\hat{a} + \hat{b}| = \sqrt{3}$.

Consider a third vector $\vec{c}$ given by $\vec{c} = \hat{a} + 2\hat{b} - 3(\hat{a} \times \hat{b})$ Then $|(\hat{a} \times \hat{b}) \times \vec{c}|^2$ is:

Q166. The value of $\sum_{n=2}^{\infty} \left(\frac{n^4 + 3n^2 + 10n + 10}{2^n (n^4 + 4)} \right)$ equals:

Q167. Let $I(a, b) = \int_a^b \sin^{-1}\left(\sqrt{\frac{2}{\pi} \sin^{-1}(\sqrt{x})} \right) dx$ Then |I(0, 1)| is

Q168. Suppose a six-faced regular die is thrown until a 6 comes up.

Let X denotes the total number of rolls. $\left|\frac{\mathrm{Var}(X)}{E(X)}\right|$ is **(Here:** E(X) and Var(X) denotes mean and variance of X)

Q169. Let $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=px^3+qx^2+rx+s$, where $p,q,r,s\in\mathbb{R}$, such that $f(1)=f(2)=0$ and $f'(1)=f'(c)=0$ such that $c\neq 1$ Then the value of c is:

Q170. $\left|\int_0^1\left[\sqrt{\frac{x+1}{x}}\right]dx\right|$ evaluates to:[ ] represents greatest integer function.

Q171. For any $x\in\mathbb{R}$, let [x] denote the greatest integer less than or equal to x.

Let the function $f:\mathbb{R}\to\{0,1,2,\ldots,9\}$ be defined by $f(x)=[10^{20}x]-10[10^{19}x]$

If $\alpha=5+\sqrt{6}$, then the value of $f(\alpha^{20})$ is

Q172. Let y(x) be a differentiable function on $[0,\infty)$ satisfying the initial condition y(0) = 0 and the equation:

$$y(x)=\frac{\int_0^x 2026t^{2025}(1-y(t))dt}{1+x^{2026}}$$

If $L=\lim_{x\to\infty} y(x)$, then the value of 100L is:

Q173. Let S be the set of points in $\mathbb{R}^3$ such that for any two points with position vector $\vec{u}$ and $\vec{v}$ in S, the point with position vector $(1-t)\vec{u}+t\vec{v}$ also lies in S for every real number t.
The set S contains points A(1, 2, 1), B(3, 0, 3) and C(-1, 2, 3).
Let C denotes the locus of all points in space $(\mathbb{R}^3)$ at a distance 5 units from the origin. Let $P=S\cap C$.
If Q denotes the projection of curve P onto xy-plane.
If the area enclosed by Q is A. Then the value of $\frac{6A^2}{\pi^2}$ is:

Q174. Let $P(x)=\sum_{k=0}^{23} a_k x^k$ be the expansion of $(1+\frac{2}{5}x)^{23}$.

Let r be the index such that $a_r=\max\{a_0,a_1,\ldots,a_{23}\}$

Let k be the index such that $|a_k(\frac{3}{2})^k|=\max\left\{|a_j(\frac{3}{2})^j|: j\in\{0,1,\ldots,23\}\right\}$

Then maximum possible sum |r + k| is:

Q 175. Let z_1 and z_2 be complex numbers constrained by sets:
$C_1=\{z_1\in\mathbb{C}:|z_1|=4\}$
$C_2=\{z_2\in\mathbb{C}:|z_2-12i|=6\}$

Define a point w in Argand plane such that it divides the segment joining z_1 and z_2 in the ratio k:1, where $k\in\mathbb{R}^+$.

Let S_k denotes the locus of w as z_1 and z_2 vary independently.

Then the maximum possible area of region S_k is:

Q176. Let T be the parabola defined by $y^2 = 4x$. Consider a variable chord PQ of T, where P and Q are distinct points on the curve.

Let $\vec{u} = \hat{i} + \hat{j}$ be a vector in the plane of parabola. If the orthogonal projection of the displacement vector $\overrightarrow{PQ}$ onto the direction of $\vec{u}$ has a constant magnitude of $10\sqrt{2}$ units, then the locus of the midpoint

M(x, y) of the chord PQ is given by: $(y^2 - 4x)(y+2)^2 + \lambda = 0$

The value of the constant λ is ______.

Q177. Let $R_{\geq 0}$ be the set of non-negative numbers. Consider a continuous function $f : R_{\geq 0} \to R$ which satisfies $f(x^2) + f(y^2) = f\left(\frac{x^2y^2 - 2xy + 1}{x^2 + 2xy + y^2}\right)$

for x, y positive real with $xy > 1$. If $f(1) = 2019$ and $f'(1) = \frac{2019}{2}$, then f(3) is:

Q178. Let S be the solid tetrahedron with boundary points

(0,0,0), (2,4,0), (5,1,0) and (3,2,10).

Let $\alpha = \max\left\{q \mid \left(\frac{12}{5}, \frac{23}{10}, q\right) \in S\right\}$

$\beta = \max\left\{\gamma \mid \left(\frac{9}{2}, \frac{5}{4}, \gamma\right) \in S\right\}$

Then $4|\alpha - \beta|$ is :

Q179. The number of functions $f:\{1,2,3,4,5\}\to\{1,2,3,4,5\}$ satisfying $f(f(f(x))) = f(f(x))$ for all x in $\{1, 2, 3, 4, 5\}$.

Q180. Let α be the real root of the equation

$$(x^3 - x - 1)\int_0^{\pi}\left(\frac{\sin^4 x}{1+e^{\pi x}+e^{-\pi x}}\right)dx = 0$$

The value of $|\alpha^{10} + 2\alpha^8 - \alpha^7 - 3\alpha^6 - 3\alpha^5 + 2\alpha^3 - 2\alpha - 17|$

.

Q181. Let $P_0(x)$ be a function defined by $P_0(x) = (x^2 - 1)^{10}$. Define a sequence of functions $\{P_k(x)\}_{k=1}^{10}$ by $P_k(x) = \frac{d}{dx}(P_{k-1}(x))$ for $k = 1,2,3,\ldots,10$

Let S denotes the set of all distinct real roots of the equation $P_{10}(x) = 0$.

Consider the disjoint intervals on $\mathbb{R}$ as: $I_1 = (-\infty,-1), I_2 = (-1,1)$ and $I_3 = (1,\infty)$

If n(A) denotes number of elements in set A, then $\sum_{k=1}^{3} n(S \cap I_k)$ is :

Q182. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x) = \int_0^x e^{x-y}f'(y)dy - (x^2 - x + 1)e^x$ Then $|f(1/2)|$ is equal to :

Q183. Let S be set of all function $f : [0, 1] \to \mathbb{R}$ satisfying:

$|f(x)-f(y)| \le |x-y|$ for all $x, y \in [0,1]$. and $\int_0^1 f(x)dx = \frac{1}{2}$

If $f, g \in S$ and $f, g : [0,1]\to\mathbb{R}$, the minimum value of $\left|6\int_0^1 f(x)g(x)dx\right|$ is:

Q184.

Let n be a positive integer and let S be a set with 2^n elements.

Let $A_1, A_2, \ldots, A_n$ be randomly and independently chosen subsets of S, where each possible subset of S is chosen with equal probability. Let P_n be the probability that: $A_1 \cup A_2 \cup \ldots \cup A_n = S$ and $A_1 \cap A_2 \cap \ldots \cap A_n = \varnothing$ Then $e^2\left(\lim_{n\to\infty} P_n\right)$ is equal to:

Q185.

The value of $\left|\sum_{n=0}^{\infty} \frac{\cos\left(\frac{n\pi}{6}\right)}{\sqrt{3}^n}\right|$ is equal to:

Q186.

Consider a Tetrahedron T formed by four planes:

$3x - 2z = 0$

$x - 7y + 4z = 0$

$x - 8y + 5z = 0$

$2y - z = 10$

Let N be the number of points with integer coordinates $(x, y, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

that are either inside or on the boundary of T. Then N is equal to:

Q187. Let F_a be family of parabola $y^2 = 4ax$, $a \in \mathbb{R}$. From an external point P tangents are drawn to F_a such that $\tan\theta + \tan\Psi = b$, where θ and Ψ are the angle made by tangents with x-axis in anticlockwise sense. Denote L(a, b) as locus of point P for a given value of a and b. $S(a,b) = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \text{ lies on } L(a,b)\}$.

If (1, 2025) $\in$ S(2024, 2025) and $(2,\beta) \in S(2024,2025)$. Then $\left|\frac{\beta - 4030}{4}\right|$ is

PARAGRAPH BASED QUESTION

Q.188 Consider two circles (disjoint) in $\mathbb{R}^2$ (x-y plane) defined as:

$C_1 : x^2 + y^2 = 1$

$C_2 : (x-3)^2 + (y-4)^2 = 4$

Let O_1, O_2 be centres of C_1 and C_2 respectively. Let L_1, L_2 be the Direct Common tangents touch C_1 at points $\{D_1, D_2\}$ and touch C_2 at points $\{D_3, D_4\}$. Let D be unique circle passing through the four points in the set $\{D_1, D_2, D_3, D_4\}$. Let its radius be R_1. Let the pair of Transverse Common tangents touch C_1 and C_2 at points $\{T_1, T_2\}$ and $\{T_3, T_4\}$ respectively. Let T be the unique circle passing through $\{T_1, T_2, T_3, T_4\}$. Let its radius be R_2. Let L (L in $\mathbb{R}^2$) denotes the line, such that length of tangent drawn from any point on L to circle C_1 and C_2 is equal. Let M be the mid point of $O_1 O_2$.

Match the Column:

(P) $4|R_1^2 - R_2^2| =$ (1) 2

(Q) If length of tangent from O_1 to circle T= α. Then $|\alpha^2| =$ (2) 6

(R) Let d(P, L) denotes perpendicular distance from a point P to the line L. Then $|5d(O_1, L)| =$ (3) 11

(S) $|20d(M, L)| =$ (4) 16

(5) 20

(A)	$(P)\to(1), (Q)\to(4), (R)\to(3), (S)\to(2)$
(B)	$(P)\to(2), (Q)\to(3), (R)\to(4), (S)\to(1)$
(C)	$(P)\to(4), (Q)\to(1), (R)\to(3), (S)\to(2)$
(D)	$(P)\to(4), (Q)\to(5), (R)\to(3), (S)\to(2)$

Q.189

Let $\vec{w} = 2\hat{i} + 2\hat{j} - \hat{k}$

Let $\vec{u}$ and $\vec{v}$ be two non-zero vectors satisfying the relation $\vec{u} \times \vec{v} = \vec{w}$ and $\vec{v} \times \vec{w} = \vec{u}$.

Consider the homogeneous system of linear equations in (x, y, z) $\in \mathbb{R}^3$ as follows:

$$x + y + z = 0$$
$$x + ty + t^2 z = 0$$
$$x + t^2 y + tz = 0$$

Let $t \in \mathbb{R} \setminus \{0,1\}$ be a value for which the System has non-trivial solutions. Match the Column below:

(P) $|\vec{v}|^2$ (1) 3

(Q) $|\vec{u} \cdot (\vec{v} \times \vec{w})|$ (2) 1

(R) $|t|$ (3) 9

(S) $|\vec{u} + \vec{w}|^2 + 9|\vec{v}|^2$ (4) 27

(5) 2

(A)	$(P) \to (2), (Q) \to (4), (R) \to (5), (S) \to (3)$
(B)	$(P) \to (1), (Q) \to (4), (R) \to (2), (S) \to (3)$
(C)	$(P) \to (5), (Q) \to (1), (R) \to (3), (S) \to (4)$
(D)	$(P) \to (2), (Q) \to (3), (R) \to (5), (S) \to (4)$

Q.190 Consider the following curves in $\mathbb{R}^2$(XY − plane).

$P: y^2 = 4x$

$C: x^2 + (y-1)^2 = 1$

$E: \frac{x^2}{4} + \frac{y^2}{2} = 1$

$H: \frac{x^2}{9} - \frac{y^2}{4} = 1$

Match the column:

(P) $S = \{(\ell, m) \in \mathbb{Z} \times \mathbb{Z} : \ell x + my + 12 = 0 \text{ is Normal to P}\}$. Then |S| is (1) 1

(Q) $T = \{(\ell, m) \in \{-1, 0, 1\} \times \mathbb{Z} : \ell x + my + 1 = 0 \text{ is Normal to C}\}$. Then |T| is (2) 2

(R) If the line x + y + k = 0 is Normal to E. Then $3k^2$ is (3) 3

(S) The number of Distinct normals that can be drawn from $(\sqrt{13}, 0)$ to H is/are (4) 4

(5) 5

(A)	$(P) \to (4), (Q) \to (3), (R) \to (2), (S) \to (1)$
(B)	$(P) \to (2), (Q) \to (4), (R) \to (2), (S) \to (3)$
(C)	$(P) \to (4), (Q) \to (1), (R) \to (3), (S) \to (2)$
(D)	$(P) \to (1), (Q) \to (3), (R) \to (4), (S) \to (5)$

Q191. Let T be the set of ordered triples $(\alpha,\beta,\gamma)\in\mathbb{R}\times\mathbb{R}\times\mathbb{Z}$. For each $w=(\alpha,\beta,\gamma)\in T$, let $y_w(x)$ be the solution of to the continuous differential equation:

$\frac{dy}{dx}+2y=\frac{\alpha\sin(\beta x)}{x^{\gamma}}, \quad x>0$ Subjected to initial condition $\lim_{x\to 0^+} y(x)=0.$

Define a function $L:T\to\mathbb{R}\cup\{\infty\}$ as: $L(w)=\lim_{x\to 0^+}\frac{y_w(x)}{x}$

Match the element $w\in T$ in **List-I** with the value L(w) in **List-II**.

List-I	**List-II**
P. w=(1,π,1)	1. 0
Q. w= (2,π,1)	2. π
R. w=(π,10,0)	3. 2π
S. w=(1,π,2)	4. π/2
	5. ∞

(A)	$P\to 1, Q\to 5, R\to 3, S\to 4$
(B)	$P\to 2, Q\to 3, R\to 1, S\to 4$
(C)	$P\to 3, Q\to 2, R\to 1, S\to 4$
(D)	$P\to 2, Q\to 1, R\to 4, S\to 5$

Q192. Let M be the set of all 3 × 3 matrices with entries from {0, 1}. Let D ⊂ M be the subset of all diagonal matrices in M. We define the set of "**Singular pairs**" S as:

$S = \{(A, B) \in M \times M : |I - AB| = 0\}$

Define the function $\phi : D \to \mathbb{Z}$ as $\phi(X) = tr(I - X)$

(Note: |Y| and tr(Y) denotes determinant and trace of matrix Y)

Match the set descriptions in **List -I** with their value(s) in **List - II**.

List-I	**List-II**
P. The value of n(S)−n(T) where $T = \{(A, B) \in M \times M : \lvert I - BA \rvert = 0\}$	1. 0
Q. Number of elements in set $\{X \in D : tr(X) = \lvert X \rvert + 2\}$	2. 7
R. Number of elements in set {X|X∈D such that (I, X)∈S}	3. 1
S. Number of elements in set {X|X∈D such that ϕ(X)=3}	4. 2^{18}
	5. 4

(A)	$P \to 1, Q \to 4, R \to 2, S \to 3$
(B)	$P \to 1, Q \to 5, R \to 3, S \to 2$
(C)	$P \to 1, Q \to 4, R \to 2, R \to 2$
(D)	$P \to 1, Q \to 5, R \to 2, S \to 3$

Q193. Let $\theta \in (0, 2\pi)$. Consider a square matrix $A(\theta)$ given by:

$$A(\theta) = \begin{bmatrix} 2\cos\theta & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let a sequence T_n be defined by:

$T_n = \text{Trace}\left((A(\theta))^n\right)$ for $n \in \mathbb{N}$.

For any $k \in \mathbb{N}$, let S_k be the set of all distinct values of $\theta \in (0, 2\pi)$ satisfying the equation: $T_k = 1$

Let n(E) denotes number of elements in Set E.

Match the set expressions in **List -I** with their corresponding values in **List - II**.

List-I	List - II
A. $n(S_3 \cup S_5)$	P. 14
B. $n(S_3 \cup S_9)$	Q. 18
C. $n(S_4 \cup S_8)$	R. 24
D. $n(S_6 \cup S_{10})$	S. 28
	T. 30

(A)	$A \to S, B \to Q, C \to R, D \to P$
(B)	$A \to P, B \to R, C \to Q, D \to S$
(C)	$A \to Q, B \to P, C \to S, D \to R$
(D)	$A \to P, B \to Q, C \to R, D \to S$

LINKED COMPREHENSION-I

Let $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 9\hat{i} + 8\hat{j}$ be the position vectors of two fixed points A and B in the cartesian plane. Consider the set Ω defined by: $\Omega = \{\vec{r} \in \mathbb{R}^2 : |\vec{r} - \vec{a}| = 14\}$

For any variable $\vec{q} \in \Omega$, let $L_{\vec{q}}$ denote the line defined by the set:

$$L_{\vec{q}} = \left\{\vec{r} \in \mathbb{R}^2 : |\vec{r} - \vec{b}| = |\vec{r} - \vec{q}|\right\}$$

Let $\vec{p}$ be the position vector of the point of intersection of the line $L_{\vec{q}}$ and the line segment connecting A and Q. As the vector $\vec{q}$ varies over the set Ω, the point $\vec{p}$ traces a conic section T.

Q194.

Consider the curve T. Which of the following(s) is (are) TRUE?	
(A)	The eccentricity of the Curve T is $\frac{5}{7}$.
(B)	The length of latus rectum of T is $\frac{24}{7}$.
(C)	$|\vec{p} - \vec{a}| + |\vec{p} - \vec{b}| = 14$ for all such $\vec{p}$
(D)	The maximum area of triangle formed with vertices $\vec{a}, \vec{b}$ and $\vec{p}$ is $10\sqrt{6}$ sq. units.

Q195.

Let θ be the angle between the vectors $\vec{p} - \vec{a}$ and $\vec{p} - \vec{b}$. Let h_1 and h_2 be the lengths of perpendiculars dropped from A and B to the line $L_{\vec{q}}$ respectively. Which of the following(s) is(are) TRUE?	
(A)	The line $L_{\vec{q}}$ is tangent to the curve T at the point $\vec{p}$.
(B)	$h_1\ h_2 = 24$ for all $\vec{q} \in \Omega$.
(C)	$|\vec{p} - \vec{a}||\vec{p} - \vec{b}|\cos^2\left(\frac{\theta}{2}\right) = 24$ holds for exactly four such $\vec{p}$ on T.
(D)	The locus of foot of perpendicular drawn from A to the line $L_{\vec{q}}$ is $|\vec{r} - (5\hat{i} + 5\hat{j})| = 7$.

LINKED COMPREHENSION-2

Consider a sequence of positive real numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ defined by $a_1 = 5$ and for $n \geq 1$, the terms satisfy the relation:

$$2a_{n+1} = 5a_n + \sqrt{21a_n^2 + 4}$$

Let $S_N = \sum_{n=1}^{N} \frac{1}{a_n a_{n+1}}$

$$S = \lim_{N \to \infty} S_N \quad \text{and} \quad \alpha = \lim_{n \to \infty} \frac{a_n}{a_{n+1}}$$

Q196.

The value of α is(are)	
(A)	$\frac{5+\sqrt{21}}{2}$
(B)	$\frac{5-\sqrt{21}}{2}$
(C)	$\frac{2}{5-\sqrt{21}}$
(D)	$5-\sqrt{21}$

Q197.

The value of Sum S is(are)	
(A)	$\frac{25-5\sqrt{21}}{2}$
(B)	$\frac{23-5\sqrt{21}}{10}$
(C)	$\alpha - \frac{1}{25}$
(D)	$\frac{1}{5} - \alpha$

LINKED COMPREHENSION-3

Let n be a positive integer and let $\binom{n}{r}$ denote the binomial coefficient $^{n}C_{r}$ for $0 \le r \le n$.

Let the sequence a_n, b_n and c_n be defined as:

$$a_n = \sum_{k=0}^{[n/3]} \binom{n}{3k} \quad b_n = \sum_{k=0}^{\left[\frac{n-1}{3}\right]} \binom{n}{3k+1} \quad c_n = \sum_{k=0}^{\left[\frac{n-2}{3}\right]} \binom{n}{3k+2}$$

Let [x] denote the greatest integer less than or equal to x. Then answer the following:

Q198.

The value of $(a_{26})^3 + (b_{26})^3 + (c_{26})^3 - 3a_{26}b_{26}c_{26}$ is	
(A)	1
(B)	2^{26}
(C)	2^{26}-1
(D)	$3 \cdot 2^{25}$

Q199.

The value of $(a_{26} - b_{26})^2 + (b_{26} - c_{26})^2 + (c_{26} - a_{26})^2$	
(A)	1
(B)	2
(C)	3
(D)	4

LINKED COMPREHENSION-4

Let C denotes the set of complex numbers. Consider the set S defined by:

$S=\{z\in C|\ z^5+z+1=0\}$

Let $T=\{w\in C|\ w=z^2-z,\forall z\in S\}$

Let |z| denote the modulus of z, and arg(z) denote the principal argument of z such that $\arg(z)\in(-\pi,\pi]$. Answer the following questions:

Q200.

The sum of all distinct elements of set T is:	
(A)	0
(B)	1
(C)	$1-\sqrt{3}\ i$
(D)	3

Q201.

The value of $\sum_{z\in S}\arg(z)$ is equal to:	
(A)	0
(B)	π
(C)	2π
(D)	$-\pi$

Q202.	
	Let $\mathbb{R}^3$ be the three-dimensional Euclidean space. Let the volume of a parallelepiped formed by any three co-terminus vectors $\vec{u},\vec{v},\vec{w}\in\mathbb{R}^3$ be defined as $V(\vec{u},\vec{v},\vec{w})=\|[\vec{u}\ \vec{v}\ \vec{w}]\|$. Let $\vec{a},\vec{b},\vec{c}\in\mathbb{R}^3$ be fixed vectors given by: $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ $\vec{b}=-\hat{i}+3\hat{j}+\hat{k}$ $\vec{c}=\hat{i}+2\hat{j}+3\hat{k}$ Define the locus set $\mathcal{L}$ for a variable vector $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ subject to the volumetric constraint: $\mathcal{L}=\left\{\vec{r}\in\mathbb{R}^3 \,\middle\|\, V(\vec{a},\vec{b},\vec{r})=\frac{9}{10}V(\vec{a},\vec{b},\vec{c})\right\}$ Which of the following statements concerning the set $\mathcal{L}$ is/are correct?
(A)	$\mathcal{L}$ constitutes a pair of parallel planes that are symmetric with respect to the origin.
(B)	The Cartesian equation representing the locus $\mathcal{L}$ is exactly $(50x + 30y - 40z)^2 = 81$.
(C)	The orthogonal distance from the plane containing the vectors $\vec{a}$ and $\vec{b}$ to any element $\vec{r}\in\mathcal{L}$ is a constant value of $\frac{9}{50\sqrt{2}}$.
(D)	The magnitude of the orthogonal projection of any vector $\vec{r}\in\mathcal{L}$ along the vector $(\vec{a}\times\vec{b})$ is exactly $\frac{9}{25\sqrt{2}}$.

Q203.	
	Consider two distinct planar regions $\mathcal{R}_1$ and $\mathcal{R}_2$ in the Cartesian plane $\mathbb{R}^2$, defined by the following sets of inequalities: $\mathcal{R}_1=\{(x,y)\in\mathbb{R}^2 \mid y^2\geq 2x,\ x^2+y^2\leq 4x,\ x\geq 0,\ y\geq 0\}$ $\mathcal{R}_2=\{(x,y)\in\mathbb{R}^2 \mid y\geq x^2-5x+4,\ x+y\geq 1,\ y\leq 0\}$ Let P be a point lying in region $\mathcal{R}_1$, and Q be a point lying in region $\mathcal{R}_2$. What is the absolute maximum possible value of the Euclidean distance PQ?
(A)	$\sqrt{17}$
(B)	$2+\sqrt{5}$
(C)	$\sqrt{13}+\sqrt{2}$
(D)	$\frac{9\sqrt{2}}{2}$

Q204.

Let $\mathcal{S} = \{L_1, L_2, \ldots, L_n\}$ be a set of n distinct lines in three-dimensional space $(n \geq 3)$. The i-th line L_i is defined by the set of points:

$$L_i = \{\vec{r} \in \mathbb{R}^3 : \vec{r} = \vec{a}_i + \lambda \vec{d}_i, \lambda \in \mathbb{R}\}$$

It is given that for every pair of indices i, j $\in \{1, 2, \ldots, n\}$ with $i \neq j$, the lines L_i and L_j intersect at a unique point. Which of the following statement(s) is/are **ALWAYS TRUE**?

(A)	If there exists a triplet of indices {i, j, k} such that $[\vec{d}_i \vec{d}_j \vec{d}_k] \neq 0$, then there exists a unique point $\vec{P}$ such that $\vec{P} \in L_m$ for all $m \in \{1, 2, \ldots, n\}$.
(B)	If the set of lines $\mathcal{S}$ is not concurrent, then for any triplet of indices {i, j, k}, the scalar triple product $[\vec{d}_i \vec{d}_j \vec{d}_k] = 0$.
(C)	If the direction vectors $\{\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_n\}$ are not all parallel to a single plane, then the lines must pass through a common point.
(D)	If n = 3 and the lines are not concurrent, then there exists a unique plane π such that $L_i \subset \pi$ for all i = 1, 2, 3.

Q205.

Let $\mathcal{E}$ be an ellipse centered at (2,3) with its major axis parallel to the x-axis, defined by

$$\frac{(x-2)^2}{a^2} + \frac{(y-3)^2}{b^2} = 1 (a > b > 0).$$

Let F_1 and F_2 be its foci. A point P on the ellipse (not on the axes) defines two focal chords PF_1Q and PF_2R. Let e be the eccentricity and S be the area of $\triangle PF_1F_2$. Which of the following statement(s) is/are **CORRECT**?

(A)	The sum of the perimeters of $\triangle PF_1R$ and $\triangle PF_2Q$ is exactly 8a, regardless of the coordinates (a, b).
(B)	If $e = \frac{1}{\sqrt{2}}$, then the area of \triangle PF_1R is equal to the area of $\triangle PF_2Q$ for all positions of P.
(C)	Let r_1, r_2, r_3, r_4 be the distances PF_1, QF_1, PF_2, RF_2 respectively. The value of $\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \left(\frac{1}{r_3} + \frac{1}{r_4}\right)$ is a constant equal to $\frac{4a}{b^2}$.
(D)	If the circle with diameter F_1F_2 is tangent to the line QR, then the point P must lie on the minor axis of the ellipse.

Q206.	Let $f : [0, 1] \to \mathbb{R}$ be a continuous and strictly increasing function such that f(0) = 0 and f(1) = 1. Let f be differentiable on (0, 1). Define the function $\Phi(x)$ for $x \in (0,1]$ as: $\Phi(x) = \frac{1}{x}\int_0^{x^2} f(\sqrt{t})dt$ If $\Phi'(x)$ denotes the derivative of $\Phi(x)$ with respect to x, then which of the following statements is/are correct?
(A)	There exists $c \in (0,1)$ such that $\Phi'(c) = 2f(c) - \frac{\Phi(c)}{c}$
(B)	There exists $c \in (0,1)$ such that $\int_0^c 2t\,f(t)\ dt = c \cdot \Phi(c)$
(C)	There exists $c \in (0,1)$) such that $f(c) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2 + k^2}$
(D)	There exists $c \in (0,1)$ such that $\Phi(c) = \frac{1}{c}\int_0^{c^2} f(\sqrt{t})\ dt$

Q207.	Let f be a strictly increasing function defined on [0, 1] such that $f \in C^2[0,1]$ with f(0) = 0 and f(1) = 1. Let a function \Phi be defined for $x \in$ (0, 1] by: $\Phi(x) = \int_0^{x^2} f(\sqrt{t})\cos\left(\frac{\pi\sqrt{t}}{2x}\right) dt$ If $\Phi'(x)$ denotes the derivative of $\Phi(x)$ with respect to x, then which of the following statements is/are correct?
(A)	There exists c (0, 1) such that $\Phi'(c) = 2cf(c)$
(B)	There exists $c \in$ (0, 1) such that $\int_0^{c^2} \frac{\pi\sqrt{t}}{2c^2} f(\sqrt{t})\sin\left(\frac{\pi\sqrt{t}}{2c}\right)dt = \frac{\Phi(c)}{c}$
(C)	There exists $c \in$ (0, 1) such that $f(c) = \int_0^1 \sin\left(\frac{\pi u}{2}\right)du$
(D)	There exists $c \in$ (0, 1) such that $\Phi(c) = \int_0^c 2tf(t)\cos\left(\frac{\pi t}{2c}\right) dt$

Q208.

Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by:

$$f(x)=\begin{cases} x^2\left|\cos\left(\dfrac{\pi}{x}\right)\right|, & x\neq 0 \\ 0, & x=0 \end{cases}$$

Which of the following statements is/are **CORRECT**?

(A)	f(x) is continuous at x = 0.
(B)	f'(0) exists and is equal to 0.
(C)	In every neighborhood $(-\delta,\delta)$ of x = 0, there exist infinitely many points where f(x) is non-differentiable.
(D)	f'(x) is continuous at x = 0.

Q209.

Consider the following two real-valued functions f(x) and g(x) defined on $\mathbb{R}$:

$$f(x)=\begin{cases} x^2, & x\in\mathbb{Q} \\ 0, & x\in\mathbb{R}\setminus\mathbb{Q} \end{cases} \quad \text{and} \quad g(x)=\sum_{n=0}^{\infty}\frac{\cos(3^n x)}{2^n}$$

Which of the following statements is/are **CORRECT**?

(A)	f(x) is continuous only at x = 0.
(B)	f(x) is differentiable at x = 0 with f'(0) = 0.
(C)	g(x) is continuous $\forall x\in\mathbb{R}$ but non-differentiable at infinitely many points.
(D)	g(x) is a nowhere differentiable function on $\mathbb{R}$.

Q210.

Let g(a) be a differentiable function defined for a \in (0, 1) by:

$$g(a)=\int_0^1 \frac{x^{a-1}+x^{-a}}{1+x}\,dx$$

Which of the following statement(s) is/are **CORRECT**?

(A)	$g(a)=\int_0^{\infty}\frac{x^{a-1}}{1+x}\,dx$
(B)	$g(a)=g(1-a)$ for all $a\in(0,1)$
(C)	$g'(1/2)=0$
(D)	$g(1/2)=\int_0^{\pi/2} 2\,d\theta$

Q211.

Let f(n) be defined for $n \in \mathbb{N}$ as:

$$f(n)=\sum_{k=1}^{n-1}\frac{\sin\left(\frac{\pi}{2n}\right)\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\cos^2\left(\frac{(k-1)\pi}{2n}\right)\cos^2\left(\frac{k\pi}{2n}\right)}$$

If $\lim_{n\to\infty}\frac{f(n)}{n^2}=\frac{a}{\pi^2}$ for $a\in\mathbb{R}^+$, which of the following statements is/are **CORRECT**?

(A)	a=4
(B)	a is the length of the latus rectum of the curve $y^2 = 4x$.
(C)	$\lim_{n\to\infty}\frac{f(n)}{n^3}=0$
(D)	The general term $T_k=\tan^2\left(\frac{k\pi}{2n}\right)-\tan^2\left(\frac{(k-1)\pi}{2n}\right)$

Q212.

Let $M_3(\mathbb{R})$ be the set of all 3 × 3 real matrices. Define a set $S \subset M_3(\mathbb{R})$ as:

$$S=\left\{A\in M_3(\mathbb{R}):\operatorname{adj}(A)=\begin{pmatrix}2 & -2 & 0\\ -6 & 9 & -1\\ 8 & -12 & 2\end{pmatrix}\right\}$$

Which of the following statements is/are **CORRECT**?

(A)	$\lvert S\rvert = 2$
(B)	$\sum_{A\in S}\operatorname{Tr}(A)=0$
(C)	$\{\det(A):A\in S\}=\{-2,2\}$
(D)	$\forall A\in S, \operatorname{adj}(\operatorname{adj}(A))=(\det A)A$

Q213.

Let S be the set of all monic cubic polynomials $P(x)=x^3+ax^2+bx+c$, where a, b, $c\in\mathbb{R}$. For a given $P\in S$, let n(P) denote the number of distinct real roots of the equation:

$$e^x = P(x)$$

If $m=\min_{P\in S}\{n(P)\}$ and $M=\max_{P\in S}\{n(P)\}$, then the ordered pair (m, M) is:

(A)	(1, 3)
(B)	(0, 4)
(C)	(1, 4)
(D)	(0, 3)

Q214.

Let V_1 and V_2 be two collections of vector pairs in $\mathbb{R}^3$ defined by: $V_1 = \{(\vec{a},\vec{b}) \in \mathbb{R}^3 \times \mathbb{R}^3 : \lvert\vec{a}\rvert^2 + \lvert\vec{b}\rvert^2 = 4\}$ $V_2 = \{(\vec{c},\vec{d}) \in \mathbb{R}^3 \times \mathbb{R}^3 : \lvert\vec{c}\rvert^2 + \lvert\vec{d}\rvert^2 = 9\}$ Let S be the set of all quadruplets $(\vec{a},\vec{b},\vec{c},\vec{d})$ such that $(\vec{a},\vec{b}) \in V_1, (\vec{c},\vec{d}) \in V_2$, and they satisfy the constraint: $(\vec{a}\cdot\vec{c}) + (\vec{b}\cdot\vec{d}) = 6$ Define a scalar function $f : S \to \mathbb{R}$ such that: $f(\vec{a},\vec{b},\vec{c},\vec{d}) = \lvert\vec{a}\times\vec{b}\rvert + \lvert\vec{c}\times\vec{d}\rvert$ Which of the following statements is/are CORRECT?	
(A)	For every element in S, there exists a constant $\lambda = \frac{3}{2}$ such that $\vec{c} = \lambda\vec{a}$ and $\vec{d} = \lambda\vec{b}$.
(B)	The function f attains its maximum value when $\vec{a} \perp \vec{b}$ and $\lvert\vec{a}\rvert = \lvert\vec{b}\rvert$.
(C)	The maximum possible value of the function f is 6.5.
(D)	For any quadruplet in S, the cross product vectors $(\vec{a}\times\vec{b})$ and $(\vec{c}\times\vec{d})$ are parallel.

Q215.

Let $\mathcal{P} \subset \mathbb{R}^2$ be the set of points defined by the parabola: $\mathcal{P} = \{(x,y) \in \mathbb{R}^2 : y^2 = 4ax, a > 0\}$ Let F(a, 0) be the focus of $\mathcal{P}$. We define a finite subset $\mathcal{V}_n \subset \mathcal{P}$ containing n distinct points $\{B_1, B_2, \ldots, B_n\}$ such that the rays $\{\vec{FB}_k\}_{k=1}^{n}$ are directed towards the vertices of a regular n-gon $(n \geq 3)$ centered at F. Let $r_k = \lvert B_k - F\rvert$ denote the focal distance and θ_k be the angle of $\vec{FB}_k$ with the positive x-axis. Which of the following propositions regarding the set $\mathcal{V}_n$ is/are CORRECT?	
(A)	The sum of the reciprocal focal distances is a constant mapping: $\sum_{B_k \in \mathcal{V}_n} \frac{1}{\lvert B_k - F\rvert} = \frac{n}{2a}$
(B)	The arithmetic mean of the focal distances in the set $\mathcal{V}_n$ satisfies: $\frac{1}{n}\sum_{k=1}^{n} r_k > 2a$
(C)	For any point $B_k \in \mathcal{V}_n$, if there exists a point $C_k \in \mathcal{P}$ such that F lies on the segment B_kC_k, then: $\frac{1}{FB_k} + \frac{1}{FC_k} = \frac{1}{a}$
(D)	Due to the rotational symmetry of the n-gon about the focus: $\sum_{k=1}^{n} \cos\theta_k = 0$

Q216. Let the locus of a point P(x, y) be defined by the quadratic form $X^T SX = 0$, where $X = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$. Let S be the augmented matrix and Q be the quadratic matrix defined as:

$$S = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \quad Q = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

Let $\Delta = \det(S)$, D = det(Q), and J = Tr(Q). Match the matrix conditions in List-I with the corresponding locus in List-II:

List-I (Conditions)	**List-II (Geometric Locus)**
(A) $\Delta \neq 0$ and D=0	(P) A Real Parabola
(B) Δ=0 and D>0	(Q) A Real Hyperbola
(C) $\Delta \neq 0$, D>0 and $J \cdot \Delta > 0$	(R) A Single Point
(D) Δ=0 and D<0	(S) An Empty Set (Imaginary)
(E) $\Delta \neq 0$, D<0 and $J \neq 0$	(T) Two Real Intersecting Lines

Q217. Consider the locus of the equation $X^T SX = 0$,

where $X = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$. Define the matrices

$$S = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \quad Q = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

Let $\Delta = \det(S)$, $D = \det(Q)$, and $J = \mathrm{Tr}(Q)$.

Match the conditions in List-I with the corresponding locus in List-II and select the CORRECT options.

List-I (Conditions)	**List-II (Geometric Locus)**
(I) $\Delta \neq 0$ and $D=0$	(P) A Real Parabola
(II) $\Delta=0$ and $D>0$	(Q) A Single Point
(III) $\Delta \neq 0, D>0$ and $J \cdot \Delta>0$	(R) An Empty Set
(IV) $\Delta=0, D=0$ and $g^2 + f^2 < Jc$	(S) Two Real Parallel Lines

Which of the following mappings are CORRECT?

(A)	(I) $\rightarrow$ (P) and (II) $\rightarrow$ (Q)
(B)	(III) $\rightarrow$ (R) and (IV) $\rightarrow$ (R)
(C)	(I) $\rightarrow$ (P) and (IV) $\rightarrow$ (S)
(D)	(II) $\rightarrow$ (Q) and (III) $\rightarrow$ (R)

Q218.

Let $S=\{x\in\mathbb{R}:1\le x\le 2\}$. Consider the functions $\phi_1,\phi_2:S\to\mathbb{R}$ defined as $\phi_1(x)=\frac{1}{\pi}\cos^{-1}(\cos\pi x)$ and $\phi_2(x)=\{x\}$, where $\{\cdot\}$ denotes the fractional part function. We define $f(x)=\max\{\phi_1(x),\phi_2(x)\}$ and $g(x)=\min\{\phi_1(x),\phi_2(x)\}$. Let the following sets represent regions in the xy-plane: $\Omega_1=\{(x,y)\in S\times\mathbb{R}:0\le y\le g(x)\}$ $\Omega_2=\{(x,y)\in S\times\mathbb{R}:g(x)\le y\le f(x)\}$ If $\text{Area}(\Omega)$ denotes the area of region \Omega, which of the following statements is/are **CORRECT**?	
(A)	There exists a value $x=c\in(1,2)$ such that $g(c)=\text{Area}(\Omega_1)$.
(B)	$\text{Area}(\Omega_2)$ is equal to $\int_1^2 \|2x-3\|dx$.
(C)	$\text{Area}(\Omega_1\cup\Omega_2)=\int_1^2 f(x)\,dx$.
(D)	The value of the ratio $\frac{\text{Area}(\Omega_1\cup\Omega_2)}{\text{Area}(\Omega_1)}$ is equal to 3.

Q219.

Let a function $f:(0,\pi)\to\mathbb{R}$ be defined by the following integral: $f(x)=3\sqrt{2}\int_0^x \frac{\sqrt{1+\cos t}}{17-8\cos t}\,dt$ Let I = f(x) and let the range of f be the set S. Suppose there exists a unique value $x_0\in(0,\pi)$ such that $\tan(f(x_0))=\frac{2}{\sqrt{3}}$. Which of the following statements is/are **TRUE**?	
(A)	The value of x_0 is $\frac{2\pi}{3}$.
(B)	f'(x) is a strictly decreasing function for all $x\in(0,\pi)$.
(C)	$f(x)=\arctan\left(\frac{4}{3}\sin\frac{x}{2}\right)$ for all $x\in(0,\pi)$
(D)	$\lim_{x\to0^+}\frac{f(x)}{\sqrt{1-\cos x}}=\frac{2\sqrt{2}}{3}$.

Q220.

Let $f:[1,\infty)\rightarrow\mathbb{R}^{+}$ be a continuous and non-increasing function. Define the infinite summation $S=\sum_{n=1}^{\infty} f(n)$ and the integral $I=\int_{1}^{\infty} f(x)\,dx$. Which of the following statements is/are **CORRECT**?	
(A)	A necessary condition for S to be a finite real number is $\lim_{n\rightarrow\infty} f(n)=0$.
(B)	If $\lim_{x\rightarrow\infty} f(x)=0$, then the integral I must always be a finite real number.
(C)	The summation S exists as a finite real number if and only if the integral I exists as a finite real number.
(D)	For the function $f(x)=\frac{1}{x(\ln x)^{p}}$ defined for x ≥ 2, the summation $\sum_{n=2}^{\infty} f(n)$ is a finite real number for all p > 1.

Q221.

Consider the ellipse E defined by the set of points: $E=\left\{(x,y)\in\mathbb{R}^{2}:\frac{x^{2}}{9}+\frac{y^{2}}{4}=1\right\}$ Let F_1 and F_2 be the foci of E where $x_{F_1}<x_{F_2}$. Let P be a point on the positive y-axis such that its distance from the origin is equal to the distance of F_1 from the origin. Two tangents L_1 and L_2 are drawn from P to the ellipse E. If \alpha is the measure of the angle between the line segment PF_1 and the tangent L_1, which of the following is/are **CORRECT**?	
(A)	The area of the quadrilateral F_1PF_2C (where C is the origin) is 5 square units.
(B)	The measure of α satisfies the relation $\tan\alpha=1/2$.
(C)	The lines PF_1 and PF_2 act as the internal and external angle bisectors of the pair of tangents $\{L_1, L_2\}$.
(D)	If m_1, m_2 are the gradients of L_1, L_2, then $m_1m_2+1/9=0$

Q222.

Consider the ellipse $E:\frac{x^2}{4}+y^2=1$ with foci $F_1(\sqrt{3},0)$ and $F_2(-\sqrt{3},0)$. Let PQ be a focal chord passing through F_1. Tangents at P and Q intersect at R. A circle S is tangent to the segment PQ and the extensions of F_2P and F_2Q. Which of the following is/are **CORRECT**?	
(A)	The circle S is tangent to PQ at the point F_1.
(B)	The center of the circle S is the point R.
(C)	RF_1 is perpendicular to the chord PQ.
(D)	If the focal chord PQ is perpendicular to the major axis, then the radius of circle S is $1/\sqrt{3}$

Q223.

Let A = (-3,0) and B = (3,0) be two fixed points in the xy-plane, and let $\mathcal{L}$ be a family of lines defined by the equation $L_m : y = mx + \sqrt{25m^2+16},$, where $m \in \mathbb{R}$ represents the slope. For any line $L \in \mathcal{L}$, let Q be an arbitrary point such that $Q \in L$. We define the distance-sum function $\Psi(Q) = QA + QB$ and the line-minimum value $\mu(L) = \min_{Q \in L} \Psi(Q)$, where QA and QB represent the Euclidean distances from Q to A and B respectively. Which of the following statements is/are **CORRECT**?	
(A)	For every line $L \in \mathcal{L}$, there exists a unique point $P \in L$ such that $\Psi(P) = \mu(L)$.
(B)	The minimum value $\mu(L)$ is invariant for all lines in the family $\mathcal{L}$ and is equal to 10.
(C)	At the unique point P where the minimum occurs, the line L is the exterior angle bisector of $\angle APB$.
(D)	For the specific line in $\mathcal{L}$ where m = 0, the minimizing point P is (0, 4).

Q224.

In the Euclidean space $\mathbb{R}^3$, let O denote the origin (0, 0, 0). Consider two sets of points S_1 and S_2 representing two lines L_1 and L_2 respectively, defined by the vector equations:

$$L_1 : \mathbf{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda\hat{i}, \quad \lambda \in \mathbb{R}$$

$$L_2 : \mathbf{r} = (-2\hat{i} + 4\hat{j} - \hat{k}) + \mu(-\hat{j}), \quad \mu \in \mathbb{R}$$

Let $\vec{a} \in S_1$ and $\vec{b} \in S_2$ be the position vectors of points A and B respectively. A real-valued function $f(\vec{a}, \vec{b})$ is defined as:

$$f(\vec{a}, \vec{b}) = |\vec{a}| + |\vec{b}| + |\vec{a} - \vec{b}|$$

Let d denote the infimum of the set of all possible values of $f(\vec{a}, \vec{b})$.

Which of the following statements is/are **CORRECT**?

(A)	The distance between the fixed reference points $P_1(1,1,-1) \in S_1$ and $P_2(-2,4,-1) \in S_2$ is $3\sqrt{2}$ units.
(B)	The value of d is $4\sqrt{2}$ units, and it is attained when A and B are collinear with the origin.
(C)	The lines L_1 and L_2 are coplanar and lie entirely in the plane $\mathbf{r}\cdot\hat{k} + 1 = 0$.
(D)	At the configuration where the minimum value d is achieved, the condition $\vec{a} \times \vec{b} = \mathbf{0}$ is satisfied.

Q225.

In the three-dimensional space $\mathbb{R}^3$, let O be the origin (0, 0, 0). Two lines L_1 and L_2 are defined by the following vector equations:

$$L_1 : \mathbf{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda\hat{i}, \quad \lambda \in \mathbb{R}$$

$$L_2 : \mathbf{r} = (-2\hat{i} + 4\hat{j} - \hat{k}) + \mu(-\hat{j}), \quad \mu \in \mathbb{R}$$

Let $\vec{a}$ and $\vec{b}$ be the position vectors of points $A \in L_1$ and $B \in L_2$ respectively. Let S be a scalar quantity defined by $S = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{a} - \vec{b}|^2$. Let d^2 be the minimum value attained by S. Which of the following statements is/are **CORRECT**?

(A)	The value of d^2 is exactly 19 units.
(B)	The points A and B that minimize S are such that $\vec{a}\cdot\vec{b} = 3.5$.
(C)	At the minimum value of S, the triangle OAB is an acute-angled triangle.
(D)	The line joining the points A and B that minimize S is parallel to the plane $z = 0$.

Q226.

Consider two sets of circles in the xy-plane defined by the following equations: $S_1 : x^2 + y^2 + 2gx - 1 = 0$ $S_2 : (x-d)^2 + y^2 + 2f(x-d) - 1 = 0$ where $g,\ f \in \mathbb{R}$ are parameters and $d > 2$ is a fixed positive integer. For a given pair of circles $C_1 \in S_1$ and $C_2 \in S_2$, let them intersect at two distinct points P and Q. A line L is a **direct common tangent** touching C_1 at A and C_2 at B. Let P be the point of intersection closer to the line L. If r_1 and r_2 are the radii of C_1 and C_2 respectively, and R is the circumradius of ΔPAB, then which of the following statement(s) is/are **CORRECT**?	
(A)	The circumradius R of ΔPAB satisfies the relation $R^2 = r_1 r_2$.
(B)	If $r_1, r_2 \in \mathbb{Z}^+$ such that $r_1 \neq r_2$, then the minimum possible value of R^2 is 2.
(C)	The length of the tangent segment AB is always equal to the diameter of the circumcircle of ΔPAB.
(D)	The circumcircle of ΔPAB passes through the midpoint of the segment AB, which lies on the common chord PQ.

Q227.

Let $H = \begin{pmatrix} 3 & 7 & 1 \\ -5 & 3 & 0 \\ 1 & 2 & 3 \end{pmatrix}$ be a 3 × 3 real matrix. Consider a unit vector $\mathbf{x} = [x_1, x_2, x_3]^T$ in $\mathbb{R}^3$ such that $x_1^2 + x_2^2 + x_3^2 = 1$. A scalar function f is defined by the quadratic form $f(\mathbf{x}) = \mathbf{x}^T H \mathbf{x}$. Which of the following statements is/are **CORRECT**?	
(A)	For any real unit vector $\mathbf{x}, f(\mathbf{x}) = \mathbf{x}^T M \mathbf{x}$ where $M = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$.
(B)	If K is the skew-symmetric part of H, then $\mathbf{x}^T K \mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}^3$.
(C)	The maximum value of the function $f(\mathbf{x})$ is 5.
(D)	The maximum value of f(x) is attained when $x_1 = x_2 = x_3 = \frac{1}{\sqrt{3}}$.

Q228.

Let $x \in \mathbb{R}$ and e be the base of the natural logarithm. Define a 2 × 2 matrix A(x) as: $A(x) = \frac{1}{2}\begin{pmatrix} e^x + e^{-x} & e^x - e^{-x} \\ -(e^x - e^{-x}) & -(e^x + e^{-x}) \end{pmatrix}$ Let $f(x) = tr(A^2(x)) - (tr(A(x)))^2$. Which of the following statement(s) is/are **CORRECT**?	
(A)	The matrix A(x) is an involutory matrix for all $x \in \mathbb{R}$.
(B)	The value of f(x) is independent of x and is equal to 2.
(C)	$\det\left(\sum_{r=1}^{n}(A(x))^{2r}\right) = n^2$ for any $n \in \mathbb{N}$
(D)	$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(A(x))^k}{k!} = (\cosh 1)I + (\sinh 1)A(x)$.

Q229.

Let C be the 3 × 3 cyclic permutation matrix defined by: $C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ Let $\mathcal{S}$ be the set of all 3 \times 3 real matrices A that satisfy the structural relation $CAC^T = A$. Which of the following statement(s) is/are **CORRECT** regarding the set $\mathcal{S}$?	
(A)	For every $A \in \mathcal{S}$, there exist scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that $A = \alpha I + \beta C + \gamma C^2$.
(B)	For any A, $B \in \mathcal{S}$, the product AB is also an element of $\mathcal{S}$ and AB = BA.
(C)	There exists a non-zero matrix $A \in \mathcal{S}$ such that the sum of all its elements is zero.
(D)	If $A \in \mathcal{S}$ is an orthogonal matrix, then there exist infinitely many such matrices.

Q230.

Let $\mathcal{F}$ be the family of all non-zero 2 × 2 real matrices A such that:

$\operatorname{tr}(A^2) = (\operatorname{tr}(A))^2$

For any matrix $A \in \mathcal{F}$, let $\operatorname{adj}(A)$ denote the adjoint of A and I be the 2 × 2 identity matrix. Which of the following statement(s) is/are **correct** for any $A \in \mathcal{F}$?

(A)	$A \cdot \operatorname{adj}(A)$ is the null matrix $\mathbf{O}$.
(B)	There exists a non-zero vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $A\mathbf{x} = \mathbf{0}$.
(C)	If $\operatorname{tr}(A) \neq 0$, then $A^n = (\operatorname{tr}(A))^{n-1} A$ for all $n \in \mathbb{N}$, $n \geq 2$.
(D)	$\det(A + I) = 1 + \operatorname{tr}(A)$.

Q231.

Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let S be the set of all 2 × 2 real matrices

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $A^T J A = J$.

Match the conditions in **Column I** with the required properties/results in

Column II.

Column I (Condition on A∈S)	**Column II (Result)**
(A) A is a symmetric matrix ($A=A^T$)	(P) ad=1 and b=c=0
(B) A is an orthogonal matrix ($A^TA=I$)	(Q) a=d and b=−c
(C) A is a diagonal matrix	(R) a=d and b=c
(D) $A^2+I=O$ (where O is the null matrix)	(S) tr(A)=0

Q232.

Let $f:\mathbb{R}\to\mathbb{R}$ be a non-constant, continuous function satisfying the functional equation: $f(2x)=2[f(x)]^2-1, \quad \forall x\in\mathbb{R}$ It is given that f(0) = 1 and f(x) possesses a global maximum value M. Which of the following statements is/are **CORRECT**?	
(A)	The global maximum M must be 1.
(B)	f(x) is an even function for all $x\in\mathbb{R}$.
(C)	f(x) satisfies the addition identity $f(x + y) + f(x - y) = 2f(x)f(y)$.
(D)	If f is twice differentiable at x = 0 with $f''(0) = -k^2$, then $f(x)=\cos(kx)$.

Q233.

Let $\mathcal{C}$ be the circle $x^2+y^2=1$ and $\mathcal{P}$ be the parabola $y = x^2 - 2$. Let $\mathcal{T}$ be the set of all lines L in the xy-plane that are tangent to the circle $\mathcal{C}$. For any line $L\in\mathcal{T}$ that intersects $\mathcal{P}$, let A(L) denote the area of the region bounded by the line L and the parabola $\mathcal{P}$. Which of the following statements is/are **CORRECT**?	
(A)	There exist exactly two lines $L_1, L_2\in\mathcal{T}$ such that $A(L_1)=A(L_2)=\min_{L\in\mathcal{T}} A(L)$.
(B)	The minimum value of the area functional A(L) is $\frac{\sqrt{3}}{2}$ sq. units.
(C)	For any $L_a, L_b\in\mathcal{T}$ with slopes $m_a = 0$ and $m_b=\sqrt{3}$, there exists a line $L_c\in\mathcal{T}$ with slope $m_c\in(0,\sqrt{3})$ such that $A'(m_c)=\frac{3\sqrt{3}-8}{6\sqrt{3}}$.
(D)	For the lines achieving the minimum area, the points of tangency on $\mathcal{C}$ are $\left(\pm\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$.

Q234.

Let S be the set of all real numbers defined by: $S=\left\{x\in\mathbb{R}:0<x<\frac{\pi}{2}\right\}$ Let $R\subset S$ be the set of all values of x that satisfy the equation: $\tan 15x=15\tan x$ If the number of elements in set R is 6 and its elements are $\{r_1,r_2,r_3,r_4,r_5,r_6\}$, then $\sum_{k=1}^{6}\frac{1}{\tan^2 r_k}$ is equal to:	
(A)	$\frac{73}{5}$
(B)	$\frac{79}{5}$
(C)	$\frac{78}{5}$
(D)	$\frac{81}{5}$

Q235.

Let $f:[0,2\pi]\to\mathbb{R}$ be defined by the determinant: $f(x)=\begin{vmatrix} 1 & \sin(2\pi\cos x) & \cos(2\pi\sin x) \\ \sin(\pi\cos x) & 1 & -\sin(\pi\cos x) \\ -1 & \sin(\pi\cos x) & 1 \end{vmatrix}$ Let $S=\{x_1,x_2,x_3,\ldots,x_k\}$ be the set of all roots of $f(x)=0$ in $[0,2\pi]$. If $a_n=\sum_{i=1}^{k}(\sin x_i)^n$ for $n\in\mathbb{N}\cup\{0\}$, then the value of infinite series $\sum_{n=1}^{\infty}a_{2n}$ is:	
(A)	$\frac{1}{3}$
(B)	$\frac{2}{3}$
(C)	1
(D)	$\frac{4}{3}$

Q236.

Let $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ for $n, r \in \mathbb{Z}$ such that $0 \le r \le n$.

Let $P(x) = (1+x)^{60} = \sum_{r=0}^{60} \binom{60}{r} x^r$.

Define $$R = \sum_{k=0}^{30} (-1)^k \binom{60}{2k} 3^k$$

$$I = \sum_{k=0}^{29} (-1)^k \binom{60}{2k+1} 3^k$$

Then which of the following is **CORRECT**?

(A)	$R + I = 2^{61}$
(B)	$R + I = 2^{30}$
(C)	$R + I = 2^{60}$
(D)	R=0

Q237.

Let F be the set of circles in the xy-plane defined by:

$$F = \left\{ C(t,k) : (x-t)^2 + (y-t)^2 + k(y-x) \mid t, k \in \mathbb{R}, k \neq 0 \right\}$$

Suppose there exists a subset of three circles $\{C_1, C_2, C_3\} \subset F$ such that each circle is externally tangent to each other, the radii $r_1 < r_2 < r_3$ of circles C_1, C_2 and C_3 respectively satisfy $(r_1, r_2, r_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

Then the minimum value of r_2 is:

(A)	4
(B)	9
(C)	16
(D)	36

Q238.

Let $L = \{ n + im \mid n, m \in \mathbb{Z} \text{ and } i = \sqrt{-1} \}$ be the set of points in the complex plane. Let S be the set of all non-zero complex numbers z such that $z + \frac{1}{z} \in L$. Let $\lvert z \rvert = r$, $r \geq 0$ and $\arg(z) = \theta, \theta \in (-\pi, \pi]$. If $\mathbb{Z}$ denotes set of integers, then which of the following(s) is (are) TRUE?	
(A)	If $e^{i\theta} \in S$ for $\theta \in (-\pi, \pi]$, there are exactly 8 possible values of θ, all of which satisfy $\sin 3\theta \cos\theta = 0$
(B)	If $z \in S$ lies on imaginary axes, then $r^2 - \frac{1}{r^2} = m\sqrt{m^2 + 4}$ for some non-zero integer m.
(C)	There exists a $z \in S$ on the ray $\theta = \frac{\pi}{6}$ such that modulus r satisfies $r + \frac{1}{r} = 2\sqrt{3}$.
(D)	If $z \in S$ then $\frac{e^{i\theta}}{\lvert z \rvert} \in S$

Q239.

Let E be the ellipse given by $E: \frac{x^2}{9} + \frac{y^2}{4} = 1$. Let P(h, k) be a point satisfying $\frac{h^2}{9} + \frac{k^2}{4} > 1$. Let L_1 and L_2 be the length of two tangents PA and PB drawn from P to the ellipse E, with the points of contact A and B. Denote: $m(P) = \min(L_1, L_2)$ $M(P) = \max(L_1, L_2)$ $\Delta(P) = \text{Area of } \Delta PAB$ Then which of the following(s) is (are) TRUE?	
(A)	$m(P) = \frac{\sqrt{77}}{4}$ for $P = (4, 0)$
(B)	$M(P) = \frac{\sqrt{70}}{3}$ for $P = (0, 3)$
(C)	$\Delta(P) = 3$ for $P = (3, 2)$
(D)	$\Delta(P) = \frac{21\sqrt{3}}{2}$ for $P = (\sqrt{11}, \sqrt{2})$

Q240.

Let $S(a_1, r)$ denotes geometric progression (GP) with first term $a_1 \in \mathbb{N}$ and common ratio $r \in \mathbb{Q} \setminus \{0\}$ which satisfies $a_1 + a_2 + a_3 = 19$ where a_k denotes k^{th} term of GP. Let F be the set of all such GP defined above. Which of the following statement(s) is (are) TRUE?	
(A)	F contains 8 such GP
(B)	$S(4, \frac{3}{2}) \in F$
(C)	There exists $S(a_1, r) \in F$ such that $a_1 a_4 = 24$
(D)	$\left\lvert \sum\limits_{S(a_1,r) \in F} a_2 \right\rvert = 38$

Q241.

Consider cube C given by: $C = \left\{ (x, y, z) \in \mathbb{R}^3 : \max\left(\left\lvert x - \frac{1}{2} \right\rvert, \left\lvert y - \frac{1}{2} \right\rvert, \left\lvert z - \frac{1}{2} \right\rvert \right) = \frac{1}{2} \right\}$ Consider family of planes P_k given by: $P_k : kx + (k-1)y + (k+1)z = 1$ Let S_k denotes the sum of squares of the length of orthogonal projections of all the edges of cube C on the plane P_k. Then which of the following(s) is(are) TRUE?	
(A)	$\sum\limits_{k=1}^{10} S_k = 80$
(B)	$P_1 \cap P_2 = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x-1}{-2} = \frac{y-1}{1} = \frac{z}{1} \right\}$
(C)	The distance d_k of the plane P_k from the origin satisfies $d_k \le \frac{1}{\sqrt{2}}$ for all $k \in \mathbb{R}$
(D)	The image of the origin (0, 0, 0) in plane P_0 is the point (0, -1, 1).

Q242. Let $S_n = \left\{ x \in \mathbb{R} : \sum_{k=0}^{n} \frac{x^k}{k!} = 0 \right\}$. If |X| denotes number of elements in set X, then the value of $|S_{20}| + |S_{26}| + |S_{27}| + |S_{33}|$ is:

Q243. Let L be a fixed line in xy-plane given by: $\sqrt{3}x + y - 6\sqrt{3} = 0$

Let a variable pair of lines defined by $x^2 + \lambda xy - y^2 = 0, \lambda \in \mathbb{R}$ intersects L at points P and Q. Let O be the origin.

Consider the following sets:

S is the set of all possible positions of CIRCUMCENTER of ΔOPQ.

G is the set of all possible positions of the CENTROID of ΔOPQ.

H is the set of all possible positions of the orthocenter of ΔOPQ.

Let A be the fixed point with coordinates (3, 4).

Define:

$d_1 = \min\{\text{dist}(A, X) : X \in S\}$
$d_2 = \min\{\text{dist}(A, Y) : Y \in G\}$
$d_3 = \min\{\text{dist}(A, Z) : Z \in H\}$

Then $[d_1^2 + d_2^2 + d_3^2]$ is:

(Note: [x] denotes greatest integer function and dist (P, Q) is distance of point P from point Q)

Q244. Let a, b, c be non zero real roots of the equation $x^3 + ax^2 + bx + c = 0$

The number of such triplets (a, b, c) $\in \mathbb{R}^3$ is (are):

Q245. Let $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ where $n \in \mathbb{Z}$ and $0 \le r \le n$. The number of integers $n \ge 10$ such that the product $\binom{n}{10}\binom{n+1}{10}$ is a perfect square is (are):.

Q246. For all natural numbers n, Let $A_n = \sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots \sqrt{2}}}}$ (n many radical).Then $\left| \frac{\lim_{n\to\infty}(2^n A_n)}{\pi} \right|$ is:

Q247. GOLU gives an exam having 20 questions. For each question the marks that can be obtained are either -1, 0 or 4. Let S be the set of all possible score GOLU can score in the examination. Then the number of elements in S is (are):

Q248. Let two rectangular hyperbolas H_1 and H_2 be defined in cartesian plane $\mathbb{R}^2$ by:

$$H_1 = \{(x, y) \in \mathbb{R}^2 : xy = k_1, k_1 \ne 0\}$$
$$H_2 = \{(x, y) \in \mathbb{R}^2 : xy = k_2, k_2 \ne 0\}$$

Let ABCD be a rhombus such that its vertices B and D lie on hyperbola H_1 and its vertices A and C lie on the hyperbola H_2. If the internal angle of the rhombus at vertex C is $\angle$ BCD = $\frac{2\pi}{3}$ and given that $k_1 \ne k_2$,

then the value of the ratio $\left|\frac{k_1}{k_2}\right|$ is

Q249.

Let $\mathbb{N}$ be set of natural numbers. Define a set of ordered pairs S as follows:

$$S = \{(A,B) \in \mathbb{N}^2 : \text{lcm}(A,B) = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7\}$$

Let n(S) denote the number of elements in S and m denotes the number of unordered pairs {A, B} such that (A, B) $\in$ S·

Then n(S) + m is

Q250.

Consider the expansion of the binomial

$$(1+x)^{2026} = \sum_{r=0}^{2026} a_r x^r.$$

Let $S = \{a_r : 0 \le r \le 2026, r \in \mathbb{Z}\}$

The number of elements in S that are not divisible by 7 is (are):

(A)	501
(B)	497
(C)	504
(D)	613

Q251.

Let S be the set of all ordered triplets of positive integer (a, b, c) such that their sum satisfies:

$$a + b + c = 17$$

Then the value of $\sum_{(a,b,c)\in S} abc$ is

(A)	$\binom{19}{3}$
(B)	$\binom{19}{5}$
(C)	$\binom{20}{3}$
(D)	$\binom{20}{5}$

Q252. Let $a, b, c, d \in \mathbb{R}$. Consider the two sets S_1 and S_2 defined by:

$$\text{Set } S_1 : \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ d \\ 0 \end{bmatrix} \right\}$$

$$\text{Set } S_2 : \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{bmatrix} 1 & 3 & -1 \\ 2 & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

If S_1 represents a straight line passing through the origin, and the intersection $S_1 \cap S_2$ contains more than one point, then the value of $|a + 2b + d + 2c|$ is:

(A)	12
(B)	14
(C)	16
(D)	18

Q253. Let S = (0, 1) be the set of parameters. For each $\alpha \in S$, define a region R_α in the xy-plane as:

$$R_\alpha = \{(x, y) \in \mathbb{R}^2, 0 \le x \le 1,\ 0 \le y \le 1,\ (1-\alpha)x + \alpha y \ge \alpha(1-\alpha)\}$$

Let Ω be the region defined by

$$\Omega = \bigcap_{\alpha \in S} R_\alpha .$$

Then the area of region Ω is:

(A)	$\frac{1}{6}$
(B)	$\frac{1}{2}$
(C)	$\frac{5}{6}$
(D)	$\frac{2}{3}$

Q254.

Let $U_n = \{z \in \mathbb{C} : z^n = 1\}$ be set of the n^{th} root of unity for $n \in \mathbb{N}$.

Define a subset of the complex plane S such that:

$$S = \left\{ z \in \mathbb{C} : \frac{1}{\sqrt{2}} \le \mathrm{Re}(z) \le \frac{\sqrt{3}}{2} \right\}.$$

Let the intersection set $I_n = S \cap U_n$.

Define $\mathrm{Arg}(z) \in (-\pi, \pi]$ as the principal argument of z and $\mathrm{Re}(z)$ is real part of z.

Let P be the smallest natural number such that $I_n \neq \phi$ for all $n > P$.

Which of the following statements is (are) TRUE?

(A)	If $z \in I_n$ then $\mathrm{Arg}(z) \in \left[-\frac{\pi}{4}, -\frac{\pi}{6}\right] \cup \left[\frac{\pi}{6}, \frac{\pi}{4}\right]$.
(B)	The minimum value of $n \in \mathbb{N}$ such that $I_n \neq \phi$ is 8.
(C)	P=16
(D)	$\lim_{n\to\infty} \frac{\lvert I_n \rvert}{n} = \frac{1}{12}$ where $\lvert X \rvert$ denote number of elements in set X.

Q255. Let the parabolic region S and the circular region D in the Cartesian plane $\mathbb{R}^2$ be defined as:

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 \le y \le 4 - x^2\}$$

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + (y-2)^2 \le 2\}$$

Define two region blue and red given by collections of points as:

$$A_b = \{(x, y) \in \mathbb{R}^2 : (x, y) \in S \cap D\}$$

$$A_r = \{(x, y) \in \mathbb{R}^2 : (x, y) \in S \text{ and } (x, y) \notin D\}$$

Let n be the maximum number of sides of a regular polygon whose all vertices lie on the boundary of S.

Let r_{max} be radius of a circle C centered on the y-axis such that $C \subseteq S$ and (0,0) lies on the circle C.

Which of the following statements is (are) TRUE?

(A)	Area of parabolic region S is $\frac{16\sqrt{2}}{3}$.
(B)	n=8
(C)	Area of region given by set $A_b = \pi - \frac{4}{3}$
(D)	$r_{max} = \frac{1}{2}$

Q256.

Let E be the ellipse given by the equation:

$$\frac{(x-1)^2}{a^2}+\frac{(y-2)^2}{b^2}=1 \text{ where } a>b>0$$

Let F_1 and F_2 be the foci of ellipse E and let e denote its eccentricity.
Suppose P is a point on E and O is the circumcenter of the triangle PF_1F_2.
If the point P satisfies the vector condition:

$$\overrightarrow{PO}\cdot\overrightarrow{F_1F_2}=2\overrightarrow{PF_1}\cdot\overrightarrow{PF_2}$$

{Note: $\vec{x}.\vec{y}$ represents dot product of vector $\vec{x}$ and y}
Which of the following statement(s) is (are) CORRECT?

(A)	$\min(e)=\sqrt{\frac{3}{8}}$
(B)	For $e=\frac{1}{\sqrt{2}}$ there are exactly two such points P on E.
(C)	$\min(e)=\sqrt{\frac{5}{8}}$
(D)	There exists some point P \in E such that circumcenter O is $\left(\frac{1}{2},\frac{1}{3}\right)$.

Q257.

Let $S=\{1,2,3,4,5\}$ and let P(S) denotes the power set of S.
Distinct subsets $A_1,A_2,\ldots,A_k$ are chosen at random from P(S).
We define the k × k symmetric matrix M as:

$$M=\begin{bmatrix}|A_1\cap A_1| & |A_1\cap A_2| & \ldots & |A_1\cap A_k|\\ |A_2\cap A_1| & |A_2\cap A_2| & \ldots & |A_2\cap A_k|\\ \vdots & \vdots & \ddots & \vdots\\ |A_k\cap A_1| & |A_k\cap A_2| & \ldots & |A_k\cap A_k|\end{bmatrix}$$

Let P_k be the probability that $\det(M)\neq 0$.
If |X| denotes number of elements in set X.
Which of the following statement(s) is (are) CORRECT?

(A)	$P_2=\frac{15}{16}$
(B)	$P_3=\frac{881}{992}$
(C)	For k = 2, the matrix M is singular if and only if $\phi\in\{A_1,A_2\}$.
(D)	$P_3=\frac{871}{992}$

Q258. Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be a function given by

$$2f(x)+2f\left(1-\frac{1}{x}\right)=3\tan^{-1}x \text{ for all real } x(\text{with } x\neq 0).$$

The value of $\frac{32\pi}{9}\left|\int_0^1 f(x)\,dx\right|$ is

Q259. Let S be set of polynomials with real coefficient defined as:

$$S=\left\{\sum_{i=0}^{2026} a_i x^i : a_i \in \mathbb{R}\right\}$$

Let $f,\ g\in S$ such that $\frac{d}{dx}(|f(x)|+g(x))=3^{\left(1+\frac{x-3}{|x-3|}\right)}\quad \forall x\neq 3$

and g(0) = f(0) = 12

Then |g(12) - f(12)| is

Q260. Let $a_i=\frac{i}{2026}$.

Then the value of $\left|\sum_{i=0}^{2026}\left(\frac{a_i^{\,3}}{1-3a_i+3a_i^{\,2}}\right)\right|$ is:

Q261.

Let $S_n = \sum_{k=1}^{n} \frac{1}{(4k+3)(4k+5)}$ for $n \in \mathbb{N}$.

And $T_n = \sum_{k=1}^{n} \frac{k^6}{n^7 + n^5k^2}$, for $n \in \mathbb{N}$

Then $\lim_{n\to\infty}\left(\frac{T_n}{S_n}\right)$ is equal to:

Q262.

Let G be the set of all infinite geometric progression defined by:

$$G = \left\{g_{a,r} : g_{a,r} = \{ar^n \mid n \in \mathbb{N}_0, a, r \in \mathbb{N}, r > 1\}\right\}$$

Where $g_{a,r}$ denotes a G.P. with first term a and common ratio r.

The number of $g_{a,r} \in G$ such that $64 \in g_{a,r}$ and $a \neq 64$ is (are)

Q263.

Let a family of lines be defined by:

$$L(\lambda) : (3x - 4y) + \lambda(3x + 4y - 7) = 0$$

where $\lambda \in \mathbb{R} \cup \{\infty\}$. Let ℓ_1 and ℓ_2 be the distinct lines in this family obtained when $\lambda \to \infty$ and $\lambda = 0$, respectively. These lines intersect at a point P.

There exists exactly four circles $\lambda \in \mathbb{R} \cup \{\infty\}$ with fixed radius $r = \frac{1}{2}$ that are tangent to both ℓ_1 and ℓ_2.

Let $O_i(x_i, y_i)$ be the center of the i^{th} circle. Let Q_i be the quadrilateral formed by vertex P, the point of tangency on ℓ_1 and ℓ_2 and the center O_i.

Define matrix $M = \begin{bmatrix} \sum_{i=1}^{4} x_i & \sum_{i=1}^{4} y_i \\ \max(\text{Area}(Q_i)) & \min(\text{Area}(Q_i)) \end{bmatrix}$

Then the Trace of M is α, then 48 α is (are)

Q264.

Denote $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$, $0 \le r \le n$ where r, n are non-negative integers.

For any positive integer n, let

$$S_n = \sum_{r=0}^{n-1} \frac{\sin(rx)\cos((n-r)x)}{\sin(nx)}$$

$$\text{Let } \sum_{r=0}^{100} \frac{\binom{100}{r}^2 (-1)^r}{r+1} = \frac{\binom{2k}{k}}{101}$$

Then S_k is equal to:

Q265.

Consider a family of circles F in complex plane such that for any circle $C_n \in F$, the locus of a point z=x+iy, $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$ on the circumference satisfies:

$\left|\dfrac{z-20\,i}{z-26\,i}\right| = t$ where $t \in \{4,5,6,7,8,9\}$

Let T be a specific circle whose centre given by $\mathrm{Re}(z_0) = \dfrac{-19}{4}$.

If N is the number of circles in family F that are orthogonal to the circle T,

then the value of N is (are)

Q266.

The number of real solutions $(x, y) \in \mathbb{R} \times \mathbb{R}$ to the system of equations

$\sin(x^2 - y) = 0$

$|x| + |y| = 2\pi$ is (are):

(A)	49
(B)	50
(C)	51
(D)	52

Q267.

Let $f(x)=\sum_{1}^{5} kx^{k-1}$ and

let $\omega=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$ where $i=\sqrt{-1}$.

Then $\prod_{n=1}^{4} f(\omega^n)$ is

(A)	144
(B)	125
(C)	169
(D)	100

Q268.

Consider a family of circles F in the xy-plane.
Each member of F is tangent to the parabola P: $x^2=10y$ at exactly two distinct points.
Let $C_1 \in F$ be a circle with radius $r_1=13$.
Let $C_2 \in F$ be another circle with radius r_2 such that C_1 and C_2 are tangent to each other externally. If S is the set of all possible r_2, then which of the following is TRUE?

(A)	$r_2=2$
(B)	$r_2=23$
(C)	Set S has two elements
(D)	$r_2=3$

Q269.

Consider the collection of all n-digit strings formed using the digits $\{1,2,3,4,5\}$. The string is defined as "IITIAN" if the digit '1' appears an even number of times, the digit '2' appears an odd number of times, the digit '3' appears at least once and there is no restriction in the frequency of digits '4' and '5'. Let a_n denote the total number of such "IITIANS" string of length n.

If $G(x)=\sum_{n=0}^{\infty}\frac{a_n}{n!}x^n$, then $G(x)=$

(A)	$\frac{1}{4}(e^{5x}-e^{4x}-e^{2x}+1)$
(B)	$\frac{1}{4}(e^{5x}-e^{4x}-e^{x+1})$
(C)	$\frac{1}{2}(e^{5x}-e^{3x}-e^{x}+1)$
(D)	$\frac{1}{4}(e^{5x}-e^{3x}-e^{x}-1)$

Q270.

Let $f(x)=\dfrac{\pi\sin(\pi x)+\ln(x+\sqrt{x^2+1})}{x^2+\cos(\pi x^3)+|x|}$ be defined for $x\in[-\pi,\pi]$.

Consider the sets A and B as follows:

$A=\{\alpha : f(\alpha)\geq f(x)\forall x\in[-\pi,\pi]\}$

$B=\{\beta : f(\beta)\leq f(x)\forall x\in[-\pi,\pi]\}$

Let $\phi(S)$ and $\Psi(S)$ denote the number of elements and sum of elements of set 'S' respectively. Then the **FALSE** statements are:

(A)	$\phi(A)-\phi(B)=0$
(B)	$\psi(A)-\psi(B)=0$
(C)	$\phi(A)+\phi(B)=3$
(D)	$\psi(A)+\psi(B)=0$

Q271.

Let L_1, L_2, L_3 be three lines in xy-plane given by:

$L_1 : x=0$

$L_2 : x+m^2-my=0$

$L_3 : my+m^2x+1=0$

Consider three points $P(m^2,2m)$, $Q(\frac{1}{m^2},\frac{-2}{m})$, $R(0,0)$.

Denote

Δ_L :area of triangle formed by lines L_1, L_2, L_3

Δ_U: area of triangle formed by points P, Q, R.

$S=\{m\in R|\ \Delta_L=3\}$

$T=\{m\in R|\ \Delta_U=4\}$

Then which of the following(s) is (are) TRUE?

(A)	$\sum_{m\in S}(\Delta_U)+\sum_{m\in T}(\Delta_L)=32$
(B)	$\|S\|=5$
(C)	$\|T\|=5$
(D)	For $m=2-\sqrt{3}$ area of triangle $\Delta PQR=2+\sqrt{3}$ sq. units.

Q272.

Consider functions $f_i : R \to R$ for i=1,2,3 as: $f_1(x) = \frac{7x}{8} + \frac{1}{2}\int_0^1 x\,t\,f_1^2(t)dt$ $f_2(x) = 1 + \frac{2}{\pi}\int_0^\pi \cos^2 x\ f_2(t)dt$ $f_3(x) = x + \int_0^1 xt\,f_3(t)\,dt$ Denote $\|f_i\|$ as the number of differentiable functions satisfying the above integral equations. Then which of the following is (are) TRUE?	
(A)	$\vert f_1 \Vert = 2$
(B)	$\Vert f_3 \Vert = 1$
(C)	$\Vert f_2 \Vert = 0$
(D)	$\Vert f_1 \Vert = 0$

Q273.

Let $S = \left\{ m \in \mathbb{N} \,\middle|\, \int_0^\pi \left(\prod_{k=1}^{m} \cos kx \right) dx = 0 \right\}$ Then which is TRUE?	
(A)	$2 \in S$
(B)	$3 \in S$
(C)	$4 \in S$
(D)	$5 \in S$

Q274.

Let $f(x)=\tan x+\cot x-2$ be restricted to its largest monotonic domain $I=\left(-\frac{\pi}{2},a\right]$ such that $f:I\to(-\infty,b]$. Let g be the inverse of f on its domain.

Define a sequence of functions $\{\phi_n\}$ such that

$\phi_1(x)=f(x)$ and $\phi_{n+1}(x)=f\left(g(\phi_n(x))+\frac{\pi}{4}+g(x)\right)$.

$\psi(x)=g(\phi_2(x))$ be defined for $x\in(-\infty,b)$.

Which of the following statement(s) is/are CORRECT?

(A)	The ordered pair $(a,b)=\left(-\frac{\pi}{4},-4\right)$.
(B)	The ordered pair $(a,b)=\left(-\frac{\pi}{6},\frac{2-\sqrt{3}}{\sqrt{3}}\right)$.
(C)	$\lim\limits_{x\to b^+}\psi'(x)=\frac{\pi}{4\sqrt{2}}$
(D)	There exists some $c\in(-\infty,b)$ such that $g'(c)=\frac{4}{\pi}\left(a+\frac{\pi}{2}\right)$.

Q275.

Let S be the set of all 3 × 3 real matrices M such that $M^T M = I$ and $\det(M) = 1$.
If for every $M\in S$ there exists a non zero vector $\vec{n}=n_1\hat{i}+n_2\hat{j}+n_3\hat{k}$ such that

$$M\begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix}=\begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix}.$$

Let $\hat{n}=[n_1n_2n_3]^T$ be corresponding unit vector and define a matrix N as:

$$N=\begin{bmatrix}0&-n_3&n_2\\n_3&0&-n_1\\-n_2&n_1&0\end{bmatrix}$$

(A)	$\lvert M^5-I\rvert=0$ for all $M\in S$.
(B)	$\lvert M+I\rvert=0$ for all $M\in S$.
(C)	The matrix M can be expressed as $M=I+(\sin\theta)N+(1-\cos\theta)N^2$, where θ is angle of rotation about $\hat{n}$.
(D)	$N^3+N=0$

Q276. Let $f:(0,\infty)\to\mathbb{R}$ be a continuously differentiable function satisfying the functional equation:

$$\int_0^{x^2} f(\sqrt{t})\,dt = \frac{x^2 f(x)}{3} + \int_0^{x} \frac{(t-t^2)^4}{1+t^2}\,dt$$

If $f(1) = \frac{22}{7} - \pi$.

If $L = \lim_{x\to 0^+} \frac{f(x)}{x^n}$ exists and is non-zero.

Then the value of $|7(n+L+4\pi)|$ is:

Q277. Let $P(x) = ax^2 + bx + c$ be a quadratic polynomial with real coefficients.

If $\sum_{n=1}^{\infty} \frac{P(n)}{2^n} = \sum_{n=1}^{\infty} \frac{P(n)}{3^n} = 2026.$

The value of b+c is:

Q278. Let A(20, 26) and B(26, 20) be two points in xy-plane.

Let line L is given by: $L: \frac{x}{47} + \frac{y}{47} = 1.$

Let S be the set of points $C \in L$ such that ΔABC is a isoscles triangle.

The number of elements in set S is(are):

Q279. Let S be the set of distinct positive roots of

$px^3 - x^2 + qx - 7 = 0$

If $(\alpha,\beta,\gamma) \in S \times S \times S$ satisfies

$$\frac{225\alpha^2}{\alpha^2+7} = \frac{144\beta^2}{\beta^2+7} = \frac{100\gamma^2}{\gamma^2+7}, \alpha \neq \beta \neq \gamma.$$

Then the value of 245P is:

Q280. Let $f:(0,\frac{\pi}{2}) \to R$ be given by

$f(x) = \cot^{-1}(2 - \tan x)$.

Define $g(\theta) = \int_{\min(\theta,\frac{\pi}{4})}^{\max(\theta,\frac{\pi}{4})} f(t)\,dt$

Then the value of $\left| \left(\frac{g(\frac{\pi}{12}) - g(\frac{\pi}{3})}{\pi^2} \right)^{-1} \right|$ is:

Q281. Consider three skew lines L_1, L_2 and L_3 in $\mathbb{R}^3$ defined by:

$L_1 : \vec{r} = 2\hat{j} - 3\hat{k} + \lambda\hat{i}, \lambda \in R$

$L_2 : \frac{x+1}{0} = \frac{y}{1} = \frac{z-3}{0}$

$L_3 : x - 1 = 0 = y + 2$

Let $S = \{(\alpha,\beta,\gamma) \in R^3\}$ be the set of all points lying on the family of lines F that intersect L_1, L_2, L_3 simultaneously.

If every point in S satisfies $\begin{vmatrix} 0 & \gamma+3 & k-\beta \\ 3-\gamma & 0 & \alpha+1 \\ \beta+2 & 1-\alpha & 0 \end{vmatrix} = 0.$

Then the value of k is:

Q282.

Let $f:(1,4)\to\mathbb{R}$ be a differentiable function such that

$f'(x)=(f(x)-\pi x)^2+\pi$ for all $x\in(1,4)$.

Let $S=\left\{\pi-\frac{1}{2},\frac{\pi}{2}+\frac{1}{20},\pi-\frac{1}{26},\pi+\frac{1}{6},\frac{\pi}{2}+\frac{1}{3}\right\}$

The number of possible value of f(3) - f(2) from the set S is / are ?

Q283.

Let $\binom{n}{r}=\frac{n!}{r!(n-r)!}$ for all $n\in\mathbb{N}, r\in W$ and $r\le n$.

Consider the sum S given by :

$$S=\binom{14}{14}\binom{20}{10}+\binom{15}{14}\binom{19}{10}+\binom{16}{14}\binom{18}{10}+\ldots+\binom{24}{14}\binom{10}{10}.$$

If $S=\binom{n}{r}$ where r, $n\in\mathbb{N}$ and n is the minimum possible value satisfying equality, then min |n-r| is/are :

Q284.

$S=\left\{x\in\mathbb{R}: x\sin x+\cos x=\int_0^2 |x-t|\,dt\right\}$

The cardinality of S is/are

(A)	0
(B)	1
(C)	2
(D)	3

Q285.

Let $A(t)=\int_0^{\infty}\frac{x^t dx}{e^x+1}$ and $B(t)=\int_0^{\infty}\frac{x^t dx}{e^x-1}$. Then $\frac{A(4)}{B(4)}$ is	
(A)	$\frac{15}{16}$
(B)	$\frac{14}{15}$
(C)	$\frac{13}{14}$
(D)	$\frac{12}{13}$

Q286.

Let $f:\mathbb{R}\to\mathbb{R}$ where $f(x)=\left(e^{x^2}+1+x+\frac{x^3}{6}-x\cos x\right)\sin x$ Let $S=\{\lambda\in\mathbb{R} \mid \text{ there exist point } c\in(0,2\pi) \text{ such that } f'(c)=\lambda f(c)\}$ Then	
(A)	$S=\mathbb{R}$
(B)	$S=\{0\}$
(C)	$S=[0,2\pi]$
(D)	S is finite set with more than 1 element.

Q287.

Let $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)=\begin{vmatrix}1 & \lvert x\rvert & x^2\\ 1 & \lvert x-1\rvert & (x-1)^2\\ 1 & \lvert x-2\rvert & (x-2)^2\end{vmatrix}+x(x-2)\lvert x^2-1\rvert\left\lvert\cos\frac{\pi x}{2}\right\rvert$ Let $S=\{k\in\mathbb{R} \mid f'(k) \text{ DOES NOT exist}\}$ and $T_r=\{x\in[0,2]: f'(x)=r\}$. Then which of the following is NOT TRUE?	
(A)	$\lvert S\rvert=3$
(B)	$\lvert S\rvert=2$
(C)	$\lvert T_0\rvert$ is NON-empty
(D)	$\lvert T_{-2}\rvert$ is NON-empty

Q288.

Let $f:\mathbb{R}\to\mathbb{R}$ be given as: $f(x)=x^3+px^2+qx+r$ where p, q, $r\in\mathbb{R}$ such that $f(1)=f(7)=f(10)=a$ and $f(2)=f(5)=f(11)=b$. Then $\lvert a-b\rvert$ is	
(A)	18
(B)	36
(C)	20
(D)	40

Q289.

A cube with edge length 1 has opposite vertex (0,0,0) and (1,1,1). Consider the points $P(p,0,0)$, $Q(1,q,0)$ $R(r,1,1)$ and $S(0,s,1)$. Then TRUE statements is/are	
(A)	If the point P, Q, R, S are coplanar then rq = (1-s)(1-p)
(B)	$\overrightarrow{PQ}=\overrightarrow{SR}$ implies centroid of quadrilateral PQRS must be at center of cube
(C)	If PQRS is a square then the area of square is $\frac{9}{8}$ sq. units.
(D)	$\overrightarrow{PQ}=\overrightarrow{SR}$ and $\lvert\overrightarrow{PQ}\rvert=\lvert\overrightarrow{PS}\rvert$ implies PQRS is a Rhombus.

Q290.

Let a, b, c be distinct reals and x, y ,z be variables such that the system of equation below can be written as AX = O where $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, then the TRUE statements is/are (Note: A is 3x3 matrix with real entry) $\begin{bmatrix}3z-2y & 2x-z & y-3x\\ y-3x & 3z-2y & 2x-z\\ 2x-z & y-3x & 3z-2y\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$	
(A)	AY=O has exactly one solution for $Y=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$
(B)	$\lvert \operatorname{adj}(\operatorname{adj}(\operatorname{adj}(2A)))\rvert=0$
(C)	$\sum_{k=1}^{10}\lvert kA\rvert=200$
(D)	$\sum_{k=1}^{10}\lvert kA\rvert=0$

Q291.

Let d(a, p) denotes shortest distance between the point (p,0) and $y^2=4ax$ for p>0 and a>0. Suppose d(a, p, b) be shortest distance between circle $(x-p)^2+y^2=b^2$ and parabola $y^2=4ax$ where a>0, p>0 and b>0. Then which is TRUE?

(A)	$d(a,p)=p \text{ if } \frac{p}{a}<2$
(B)	$d(a,p)=2\sqrt{a(p-a)} \text{ if } \frac{p}{a}\geq 2$
(C)	$d(a,p,b)=0 \text{ for } \frac{p}{a}<2 \text{ and } p>b$
(D)	$d(a,p,b)=2\sqrt{ap-a^2}-b \text{ if } 4a(p-a)>b^2 \text{ and } \frac{p}{a}\geq 2$

Q292

Consider Family of Lines L_p given by

$L_p:(1+p)x-py+p(1+p)=0$

Let $H(L_A,L_B,L_C)$ denotes orthocenter of triangle formed by lines L_A,L_B and L_C.

Then which of the following statement(s) is (are) TRUE?

(A)	$H(L_{20},L_{26},L_{(-1)})$ lies on x+y=0.
(B)	$H(L_{20},L_{26},L_{(-1)})$ lies on x+3y=0.
(C)	$H(L_{21},L_{27},L_{(-1)})$ lies on x+y=0.
(D)	$H(L_{21},L_{27},L_{(-1)})$ lies on x+3y=0.

Q293.

Let S be the set of eight position vectors defined by: $S=\left\{x\hat{i}+y\hat{j}+z\hat{k}: x,y,z\in\{-1,1\}\right\}$. Three distinct vectors are chosen randomly from set 'S' without replacement. Let V denote the volume of the parallelopiped formed by coterminous edges $\vec{u},\vec{v},\vec{w}$ chosen from set S. Which of the following is/are CORRECT?	
(A)	The number of ways to choose the vectors such that they are coplanar is 24.
(B)	The set of all possible values of V is $\{0, 4\}$.
(C)	The probability that the vectors $\vec{u},\vec{v},\vec{w}\in S$ forms linearly independent set is $\frac{4}{7}$.
(D)	There exists $\vec{u},\vec{v},\vec{w}$ (all distinct) vectors in S such that $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{w}=\vec{u}\cdot\vec{w}=0$

Q294.

$S=\left\{(x,y)\in\mathbb{N}\times\mathbb{N}: \tan^{-1}\left(\frac{1}{x}\right)+\tan^{-1}\left(\frac{1}{y}\right)=\tan^{-1}\frac{1}{7}\right\}$ Then which of the following(s) is(are) true? (Note : |x| denotes cardinality of set S)	
(A)	$(8,57)\in S$
(B)	$(17,12)\in S$
(C)	$\|S\|=8$
(D)	$\|S\|=6$

Q295.

Let $\mathbb{C}$ denotes the set of complex numbers.
Let $S=\{z\in\mathbb{C}: |z^2-3z+3|=1\}$
Then $\min_{z\in S}\left|z-\frac{3}{2}\right|+\max_{z\in S}|z-2|$ is

Q296.

Let f be a continuous function $\mathbb{R}^+ \to \mathbb{R}^+$, define

A_r = Area bounded by graph of f, x-axis, x = 1 and x = r.

B_r = Area bounded by graph of f, x-axis, x = r and x = r^2.

$S = \{f(x) : A_r = B_r \text{ for all } r \in \mathbb{R}^+\}$.

The number of elements in set S, such that f(1) = 2 is (are)

Q297.

Consider circle S given by set $\{(x, y) \in \mathbb{R}^2 | \ x^2 + y^2 = 4\}$.

Denote family of lines L_m as $(y - \sqrt{2}) = m(x - 1)$.

Suppose L_a and L_b intersects circle S at points A, C and B, D respectively.

If $L_a \in L_m$ and $L_b \in L_m$ such that L_a is perpendicular to L_b. Then maximum area of quadrilateral ABCD equals:

Q298.

Let $f : [0,2] \to [0,1]$ be a function defined by:

$$f(x) = \left(\frac{x}{2} + \sqrt{\frac{x^2}{4} + \frac{1}{27}}\right)^{\frac{1}{3}} + \left(\frac{x}{2} - \sqrt{\frac{x^2}{4} + \frac{1}{27}}\right)^{\frac{1}{3}}.$$

Define $g(x) = \{(b, a) \in [0,1] \times [0,2] | \ f(a) = b\}$ that is $g(x) = f^{-1}(x)$.

Let A(c) denote the area of the region bounded by y = g(x) and the tangent drawn at x = c to the graph of g(x) and the lines x = 0 and x = 1 for any point c $\in$ [0, 1].

There exists a unique point $\lambda \in [0,1]$ such that $f(\lambda) \le f(c)$ for all $c \in [0,1]$.

Then λ is:

Q299. Let C be the curve in xy-plane given by:

$$C = \{(x, y) \in \mathbb{R}^2 | \ y = x^4 + x^3 - x^2 + 2x\}$$

Let S denotes set of Lines passing through origin and tangent to curve C.

Then number of elements in set S is(are) :

Q300. If the number of ordered pairs $(m, n) \in \mathbb{Z}^2$ such that for all non-negative integers $k \in \{0, 1, 2, \ldots\}$, we have $0 \le m < 2026,\ n \ge 0$ and $n - m \le k(k - 2n + 1)$ is N. Then sum of all digits of number N is(are):

Q301. Let $P_k(n)$ denotes sum of k^{th} power of first n natural numbers.

Suppose we define a polynomial in x corresponding to $P_k(n)$ as $P_k(x)$.

For example $P_3(n) = \sum_1^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$, So we have $P_3(x) = \left(\frac{x(x+1)}{2}\right)^2 \ \forall x \in \mathbb{R}$

(Corresponding polynomial). Then $|P_7(-3) + P_6(-4)|$ evaluates to :

(A)	664
(B)	665
(C)	666
(D)	667

Q302.

Let $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)=x^3+x$, and Let $g:\mathbb{R}\to\mathbb{R}$ be such that $g(f(x))=f(g(x))=x\forall x\in\mathbb{R}$. Then $\lim\limits_{x\to\infty}\dfrac{f(x)^{\frac{1}{9}}+g(x)}{(x)^{\frac{1}{3}}}$ is:	
(A)	0
(B)	2
(C)	½
(D)	1

Q303.

Let the family of function f_n be defined as $f_1(x)=\|x^3-x^2\|$ and $f_{k+1}(x)=\|f_k(x)-x^3\|$ for all $k\in\mathbb{N}$. Define $T=\{x\in\mathbb{R}\| f_n(x)$ is NOT differentiable in first quadrant$\}$ $S_n=\{\sum f_n(x)\| x\in T\}$. Then $\lim\limits_{k\to\infty}\left(\dfrac{S_{2k+1}}{S_{2k}}\right)$ is	
(A)	$\frac{1}{7}$
(B)	$\frac{1}{6}$
(C)	7
(D)	6

Q304.

Let S be the sphere in 3-D space given by:

$$S=\{(x,y,z)\in\mathbb{R}^3 \mid (x+2)^2+(y-3)^2+(z-4)^2=24\}$$

Let $O_{i,j,k}$ denotes octant given by:

$$O_{i,j,k}=\{(x,y,z)\in\mathbb{R}^3 \mid \operatorname{sgn}(x)=i, \operatorname{sgn}(y)=j, \operatorname{sgn}(z)=k\}$$

where $i,j,k\in\{-1,1\}$ and $\operatorname{sgn}(x)=\begin{cases}1, & x\geq 0\\ -1, & x<0\end{cases}$

Then the number of octant $O_{i,j,k}$ such that $S\cap O_{i,j,k}\neq\phi$ is (are)

(A)	5
(B)	6
(C)	7
(D)	8

Q305.

Let $t\in\mathbb{R}$ and $f_t:\mathbb{R}\to\mathbb{R}$ given by $f_t(x)=x+te^{-x^2}$ for all $x\in\mathbb{R}$. Then the true statement is/are

(A)	$S=\{t\in\mathbb{R} \mid f_t(x)$ surjective$\}$ then S has infinite element.
(B)	$f_{20}(x)$ is surjective.
(C)	f_t is bijective if and only if $\lvert t\rvert\leq\sqrt{\frac{e}{2}}$
(D)	$f_{25}(x)$ is bijection.

Q306.

Let $S=\left\{t\in\mathbb{R}:f(x)=|x-\pi|\left(e^{|x|}-1\right)\sin|x|\text{ is not differentiable at }x=t\right\}$.

$$T=\begin{Bmatrix}(\lambda,\mu)\in\mathbb{R}\times\mathbb{R}:f(x)=\left(|\lambda|e^{|x|}-\mu\right)\sin(2|x|),\\ f(x)\text{ is a differentiable function for all }x\in\mathbb{R}\end{Bmatrix}$$

Then Which of the following(s) is (are) TRUE?

(A)	$T\subseteq\mathbb{R}\times[0,\infty)$
(B)	$S=\phi$
(C)	$S=\{0,\pi\}$
(D)	$T\subseteq[0,\infty)\times\mathbb{R}$

Q307.

Let $f:(0,1)\to\mathbb{R}$ be given by:

Let $f(x)=\cos\left(2\tan^{-1}\left(\sin\left(\cot^{-1}\left(\sqrt{\frac{1-x}{x}}\right)\right)\right)\right)$ and

$$S=\left\{x\in\mathbb{R}|\ \tan^{-1}\sqrt{x^2+x}+\sin^{-1}\sqrt{1+x+x^2}-\cos^{-1}(1-2x^2)=\frac{\pi}{2}\right\}$$

{Note: $f'(x)=\dfrac{df}{dx}$ and |X| denotes set X.} Then which is TRUE?

(A)	$(1-x)^2f'(x)+2(f(x))^2=0$
(B)	$\lvert S\rvert=0$
(C)	$(1+x)^2f'(x)-2(f(x))^2=0$
(D)	$\lvert S\rvert=1$

Q308.

From a point P(4,5) tangents PQ and PR are drawn to circle $x^2 + y^2 + 2x = 0$ centered at C such that Q and R are points of contact of tangents drawn. Suppose PC meets the circle in points A and B such that PA > PB. Let S be the point of intersection of chord QR and line PC.

Then which of the following(s) is (are) TRUE?

(Note: A.M, G.M, H.M denotes Arithmetic, Geometric and Harmonic means respectively)

(A)	CP is A.M. of AP and BP
(B)	PR is G.M. of PS and PC
(C)	PS is H.M. of PA and PB
(D)	$PA \times PB = PS \times PR$

Q309.

Let $P:\mathbb{R}\to\mathbb{R}$ be such $P(x)=1+x+\frac{x^2}{2!}+\ldots+\frac{x^{2026}}{2026!}$.

Let f be twice differentiable function on $\mathbb{R}$ such

that $f''(x)+P(x)f'(x)-f(x)=0\ \forall x\in\mathbb{R}$.

If $f(x)=\sin(\pi x)\left(ax^3+be^x+c\cos(\pi e^x)\right)$ for a, b, $c\in\mathbb{R}$

Then which of the following (s) is (are) CORRECT?

(A)	There exists some $c\in(0,1)$ such that $\int_0^1 e^x f(x)\,dx=c$
(B)	There exists some $c\in(0,1)$ such that $\int_0^1 e^x f(x)\sin(\pi x)=f(c)$
(C)	$\frac{a}{2}+b\pi=c$
(D)	The set $\left\{x\in(0,1) \mid 2^x-x^2=\int_0^1 f(f(x))\,dx\right\}$ has has two elements

Q310. Let S denotes the set of complex numbers. For any $\omega \in S$, let L_ω be a line in $\mathbb{R}^3$ defined by the locus of points (x, y, z) such that:

$\frac{x+iy}{1+iz} = \omega$, where $i = \sqrt{-1}$

If $S = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} = \{-1, 1, e^{i\frac{2\pi}{3}}, e^{i\frac{\pi}{4}}, e^{-i\frac{\pi}{18}}\}$

Let $d(L_{\omega_m}, L_{\omega_n})$ denotes the shortest distance between two lines L_{ω_m} and L_{ω_n} for $m \neq n$. Then which of the following(s) is(are) CORRECT?

(A)	For any distinct $\omega_m, \omega_n \in S$, the shortest distance $d(L_{\omega_m}, L_{\omega_n}) = \lvert \omega_m - \omega_n \rvert \sin^2(\theta)$ where θ is acute angle between the lines.
(B)	$\lvert L_{\omega_5} \cap L_{\omega_3} \rvert = 0$, where $\omega_5 = e^{-i\frac{\pi}{18}}$ and $\omega_3 = e^{i\frac{2\pi}{3}}$. { |X| { denotes number of elements in set X }
(C)	If θ is the angle between two lines L_{ω_m} and L_{ω_n} then $\cos\theta = \frac{1+\mathrm{Re}(\omega_m \overline{\omega_n})}{2}$
(D)	The number of skew pairs $(L_{\omega_m}, L_{\omega_n})$ in as $\omega_i \in S$ for $i \in \{1,2,3,4,5\}$ is 8.

Q311. Let A.P. (a, b, n) denotes Arithmetic progression with first term, common-difference and number of terms as a, b and n respectively. Two numbers m and k such that $1 < m < n$ and $1 < k < n$ where $n \in \mathbb{N}$. are removed from A.P. (1, 1, n) and the average of remaining number is found to be 17. Then the maximum possible sum of m+k is

Q312. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \in \{-1, 1\}\}$. Suppose Q be the Cube with set of vertices from Set 'S'. The number of distinct invertible 3 × 3 matrix M such that for every (x, y, z) $\in S$ we have (u, v, w) $\in S$ and $M\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ is (are):

Q313. Let $(1+\sqrt{2}i)^n = s_n + it_n$ where $s_n, t_n \in R, i = \sqrt{-1}$.

Then the value of $\frac{2s_{11} - s_{12}}{s_{10}}$ equals :

Q314. Let C be the locus of image of point $(\frac{1}{\sqrt{2}},0)$ in the variable line $\sqrt{2}y=\sqrt{2}mx+\sqrt{1+2m^2}$. Tangent drawn from the point $P(\frac{1}{\sqrt{2}},\sqrt{6})$ meets curve C in A and B. The distance between incenter and orthocenter of ΔPAB is $2(\sqrt{k}-1),k\in\mathbb{N}$. Then k is

Q315. Let P be the probability of getting head on a toss of coin. The value of P such that it maximizes the probability of getting 6 heads when coin is flipped 10 times is α.

Then 10α is:

Q316. Let a sequence of n=25 independent Bernoulli trials be represented by

$S=\{s_1,s_2,s_3,\ldots,s_{25}\}$, where each trial $s_i\in\{H,T\}$ with $P(s_i=H)=\frac{1}{2}$.

Define a random variable X as total number of occurrences of the consecutive doublet HH within S. Specifically, X counts overlapping pairs such that for any subsequence s_i,s_{i+1} a success is recorded if $s_i=s_{i+1}=H$.

Then the expectation E(X) is

ANSWER KEY

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	D	C	C	B	D	C	A	A	B
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	C	C	A	B	B	B	C	A	D	B
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	C	C	B	B	D	B	C	D	B	A
Que.	31	32	33	34	35	36	37	38	39	40
Ans.	B	C	A	C	D	B	C	B	C	B
Que.	41	42	43	44	45	46	47	48	49	50
Ans.	B	ABCD	ABC	ABC	ACD	ABC	ABC	ABC	AC	ABC
Que.	51	52	53	54	55	56	57	58	59	60
Ans.	ABD	AC	ABD	ABC	AC	BCD	ACD	ABCD	AC	ABCD
Que.	61	62	63	64	65	66	67	68	69	70
Ans.	AC	ABCD	ACD	ABCD	A	ABC	ABD	ABC	ABD	BCD
Que.	71	72	73	74	75	76	77	78	79	80
Ans.	A	ABCD	AD	AD	AB	ACD	BCD	AD	ABCD	ABD
Que.	81	82	83	84	85	86	87	88	89	90
Ans.	ACD	ABCD	ABCD	A	ACD	ABD	ACD	ABD	AB	ACD
Que.	91	92	93	94	95	96	97	98	99	100
Ans.	ACD	ABC	6	6	144	12	360	8	6	2
Que.	101	102	103	104	105	106	107	108	109	110
Ans.	1	0.5	2	2026	2026	15	134	10	512	642
Que.	111	112	113	114	115	116	117	118	119	120
Ans.	0	2	60	20	32	4	4	18	9	1

Que.	121	122	123	124	125	126	127	128	129	130
Ans.	18	2	722	10	530.10 to 530.30	100	4	6	21	33
Que.	131	132	133	134	135	136	137	138	139	140
Ans.	899	18	25	4	26	50	10	19	24	9
Que.	141	142	143	144	145	146	147	148	149	150
Ans.	1	576	6	15	70	3	20	3	10	8
Que.	151	152	153	154	155	156	157	158	159	160
Ans.	5	6	16	240	3	10	229	4	3	2
Que.	161	162	163	164	165	166	167	168	169	170
Ans.	7	202	0	5	$\frac{21}{4}$	$\frac{11}{10}$	0.75-0.80	5	1.66 - 1.67	$\frac{7}{4}$
Que.	171	172	173	174	175	176	177	178	179	180
Ans.	9	50	361	15	75.2-75.6	400	673	15	756	15
Que.	181	182	183	184	185	186	187	188	189	190
Ans.	10	0	1	1	1.5	286	5	C	D	A
Que.	191	192	193	194	195	196	197	198	199	200
Ans.	B	D	D	ACD	ABD	B	B	B	B	A
Que.	201	202	203	204	205	206	207	208	209	210
Ans.	B	ABC	B	ABCD	ACD	ABCD	CD	ABC	ABCD	ABCD
Que.	211	212	213	214	215	216	217	218	219	220
Ans.	ABCD	ABCD	B	ABCD	ABCD	(A) $\to$ P, (B) $\to$ R, (C) $\to$ S, (D) $\to$ T, (E) $\to$ Q.	ABD	ABCD	ABCD	ACD
Que.	221	222	223	224	225	226	227	228	229	230
Ans.	ABCD	ABCD	ABCD	ABCD	CD	AB	ABCD	ABCD	ABCD	ABCD
Que.	231	232	233	234	235	236	237	238	239	240

Ans.	$(A) \rightarrow R$, $(B) \rightarrow Q$,	ABCD	ABCD	B	D	C	B	ABD	ABC	BC
Que.	241	242	243	244	245	246	247	248	249	250
Ans.	ACD	2	26	2	1	1	95	3	1418	C
Que.	251	252	253	254	255	256	257	258	259	260
Ans.	B	B	C	ABCD	ABD	AB	ABC	2	108	1013.5
Que.	261	262	263	264	265	266	267	268	269	270
Ans.	2	14	233	24.5	6	D	B	B	A	BC
Que.	271	272	273	274	275	276	277	278	279	280
Ans.	A	ABC	AD	AD	2	120	6078	5	9	144
Que.	281	282	283	284	285	286	287	288	289	290
Ans.	2	1	10	C	A	A	A	D	ABCD	ABD
Que.	291	292	293	294	295	296	297	298	299	300
Ans.	AC	ABC	ABD	ABC	$\frac{5}{2}$	1	5	$\frac{1}{2}$	3	16
Que.	301	302	303	304	305	306	307	308	309	310
Ans.	B	B	A	B	ABC	ABD	AD	ABC	BC	BC
Que.	311	312	313	314	315	316				
Ans.	51	48	3	2	6	6				

Q.1

$$\int_0^1\left((1-x^n)^{\frac{1}{n}}-x\right)^2 dx=\int_0^1(1-x^n)^{\frac{2}{n}}dx-2\int_0^1 x(1-x^n)^{\frac{1}{n}}dx+\int_0^1 x^2dx$$

M.2 $\int_0^1 x(1-x^n)^{\frac{1}{n}}dx$

$=\int_0^1 xtdx$ let $(1-x^n)^{\frac{1}{n}}=t$ (Symm)

$=t\frac{x^2}{2}\Big|_0^1+\int_0^1(\frac{x^2}{2}(\frac{dt}{dx}))dx$ $\Rightarrow 1-x^n=t^n$

$=\frac{1}{2}\int_0^1 x^2dt=\frac{1}{2}\int_0^1(1-t^n)^{\frac{2}{n}}dt$ $t=(1-x^n)^{\frac{1}{n}}$ Inves $x=(1-t^n)^{\frac{1}{n}}$

M.3 $\int_0^1(f(x))^2-2\int_0^1 xf(x)dx$

$\int_0^1 xf^{-1}(x)dx$

$\int_0^1 tf(t)f'(t)dt$

$=t(f^2(t)/2)\Big|_0^1-\int_0^1\frac{f^2(t)}{2}dt=0$

M.3 $\int_a^b f(x)dx+\int_{f(a)}^{f(b)}f^{-1}(x)dx=bf(b)-af(x)$

Put $x=(\sin\theta)^{2/n}$ using Wallis we can do (not effective though)

Q.2

Eq of chord with $M(x_1, \sqrt{x_1})$ is $T=S_1$

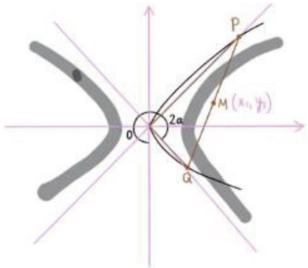
$$2x - y\sqrt{x_1} = x_1$$

Homogenize $y^2=4x$ with chord.

$$x_1y^2 = 8x^2 - 4xy\sqrt{x_1}$$

$$\Rightarrow 8x^2 - 4\sqrt{x_1}xy - x_1y^2 = 0$$

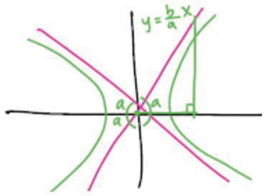
Let 2α be angle between asymptotes.



$$\tan(2\alpha) = \frac{2\sqrt{(4\sqrt{x_1})^2 - 8(-x_1)}}{|8-(-x_1)|} = \frac{4\sqrt{3x_1}}{|8+x_1|}$$

Since, $\sec\alpha = e$

$$\Rightarrow \sec\alpha\tan\alpha\frac{d\alpha}{dx_1} = \frac{de}{dx_1} \quad (\text{at } x_1 = 8, \frac{de}{dx_1} = \text{NOT defined})$$



$$\tan\alpha = \frac{b}{a} \Rightarrow 1+\tan^2\alpha = \frac{a^2+b^2}{a^2} = e^2 \boxed{\Rightarrow \sec\alpha = e}$$

Q.3

$$\det(M) = z(z+1)-(1)(-1) = z^2+z+1 \in \mathbb{R}$$

$$\Rightarrow \text{Re}(z) = -\frac{1}{2} \quad (\text{Take conjugate})$$

$$|\alpha|^2 = \frac{|z_0|^2 - r^2 - 1}{|z_0|^2 - 4r^2 + 1}$$

$$= \frac{(1-2r^2)-r^2-1}{(1-2r^2)-4r^2+1} = \frac{1}{2}$$

Q4. $f(0)=1 \rightarrow (0,0,0)$

$f(1)=2 \rightarrow (0,1,0),(0,0,1)$

$f(2)=4$ etc.

$$\sum_{n=0}^{\infty}\frac{f(n)}{2^n}=\left(\sum_{a\geq 0}\frac{1}{2^{2a}}\right)\times\left(\sum_{b\geq 0}\frac{1}{2^b}\right)\times\left(\sum_{c\geq 0}\frac{1}{2^c}\right)$$

BONUS QUESTION

Let N be the set of all non-negative integers. Solutions (x, y, z) to the equation x+y+z=10. A solution (x, y, z) is chosen uniformly at random from N. Let A={1,2,….,10}. Based on chosen values of (x, y, z) consider the following sequence of process of choosing subsets of A. A subset S_1 of size x is chosen from A. A subset A\S1. A subset S3 of size z is chosen from the set $S_1 \cup S_2$. Let E be the even that it is possible to make those choices such that S3=S1. Let F be the vent that Z is an even integer.

The Conditional probability P(F|E) is:

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{3}{5}$ (D) $\frac{2}{3}$

SOL.

For $S_1 = S_3 \Rightarrow |S_1| = |S_3| \Rightarrow \boxed{x=z}$

We chose S_1(size x) and S_2(size y) from disjoint pools.

S_3 must be chosen from $S_1 \cup S_2$. Since S_1 is a subset of $S_1 \cup S_2$, we can choose S_1 as our candidate for S_3 which provides size requirement (z) matches S_1.

Therefore z=x is necessary and sufficient. Since $|S_1 \cup S_2| = x+y$. We need

$x+y \geq z \Rightarrow x+y \geq x \Rightarrow y \geq 0$ (No case eliminated)

So, $x+y+z=10$ subject to $x=z$ $(x, y, z \geq 0)$

$\Rightarrow \boxed{2x+y=10}$

Total valid outcomes for event E=6

$|F \cap E| = 3$ (check even is 3)

$$P(F\,|\,E)=\frac{|F\cap E|}{|E|}=\frac{3}{6}=\frac{1}{2}$$

Q.5

$\bar{N} = N - S_1 + S_2 - S_3 + S_4$ (Where $A_S, A_I, A_M, A_P \rightarrow$ four properties)

$S_1 = \binom{4}{1} \times 5! = 480$

$S_2 = \binom{4}{2} \times 4! = 144$

$S_3 = \binom{4}{3} \times 3! = 24$

$S_4 = \binom{4}{4} \times 2! = 2$

$\bar{N} = 720 - 480 + 144 - 24 + 2 = 362$

Q.6

$|A_1| = |A_2| = |A_3| = 1$ and $|A_4| = 3$.

Let's check options to reject:

$|M_A| = 9 = 3^2$ (possible)

$|M_B| = 1 = 3^0$ (possible)

$|M_C| = 9 = 3^2$ (possible)

$|M_D| = 1 = 3^0$ (possible)

So, det is not discriminating

Observe:

$M_B = A_1^3$

$M_A = A_4^2$

$M_C = A_4\left(A_1A_4A_1^2\right)\left(A_3^5 \cdot A_2\right)$

M_D= cannot be obtained (why?)

$$\text{Logic: } M_C = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\textbf{NOT available}} \underbrace{\begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Rotation matrix (30° rotation about z-axis)}}$$

$$A_4 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{So: } \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A_4 \cdot (A_1A_4A_1^2)$$

Now $A_3^5A_2$ does the remaining work.

$180^\circ - 150^\circ = 30^\circ$

$$M_D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \rightarrow \text{rotation (pure) about y-axis by } \pi/2$$

<u>Available rotation in set 'S'is</u>

A_3: 60° about z-axis

A_2: 180° about z-axis (with sign flip)

A_1: 120° rotation about axis (1, 1, 1).

With the help of these we cannot obtain A '90°' rotation as all orthogonal matrix

gives only 60° and 120° in specific combination here.

Q.7

$d_1 \equiv (\text{OA and BC})$

$d_2 \equiv (\text{OB and AC})$

$d_3 \equiv (\text{OC and AB})$

$$d_1 = \frac{bc}{\sqrt{b^2+c^2}}$$

$$\Rightarrow \frac{1}{d_1^2} = \frac{1}{b^2} + \frac{1}{c^2} \quad \text{(use skew line result)}$$

$$\Rightarrow \frac{2}{a^2} + \frac{2}{b^2} + \frac{2}{c^2} = \frac{3}{4}$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{3}{8}$$

$a \le b \le c \to \text{W.L.O.G} \to a = 1$ X

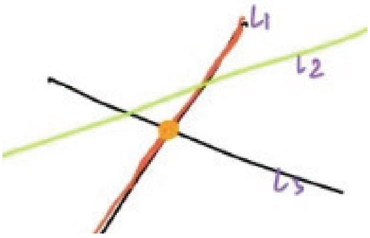
$a = 2 \Rightarrow c = 4, b = 4$

So, $(a, b, c) \in \{(2,4,4), (4,2,4), (4,4,2)\}$

Q.8

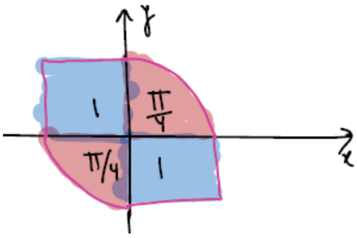
$C_1 : (x+y)(x-y+1) = 0$

$C_2 : (ax+y)(x+y+2) = 0$



$|C_1 \cap C_2| \to \text{for } \infty a = 1$ as x+y=0 is common line to both family.

Q.9



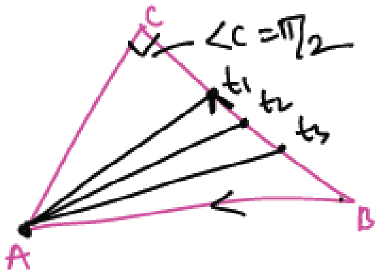
use : $\sin^{-1} x = \theta_1, \theta_1 \in [-\pi/2, \pi/2]$

$\sin^{-1} y = \theta_2, \theta_2 \in [-\pi/2, \pi/2]$ (analyse in each quadrant)

(Note : unique question created to revise full chapter elegantly!)

Q.10

$|\overrightarrow{BA} - t\overrightarrow{BC}| \geq |\overrightarrow{AC}|$



As true for all $t \in R$

$\overrightarrow{BC} \cdot \overrightarrow{AC} = 0$

$(1, c-b) \cdot (a-3, 1-c) = 0$

$\boxed{a - 3 + (1-c)(c-b) = 0}$

$\Rightarrow \boxed{b = c + \frac{3-a}{c-1}}$

$(3-a) \in \{-2, -1, 0, 1, 2\}$

Total 14 cases out of which (3, 1, 1), (3, 2, 1), (3, 3, 1), (3, 4, 1), (3, 5, 1) are degenerate! (Source: "Thin book with Fat Ideas" book by Kunal Thakur)

Q.11

Let $P_A = x_1x_4$ and $P_B = x_3x_2$,

$P(\text{product } 1) = P(X_1 = 1) \times P(X_4 = 1) = 4/9$

$P(\text{product } 0) = 1 - P(\text{Probability of product } 1) = 5/9$

Now $Z = X_1X_4 - X_3X_2 \in \{-1, 0, 1\}$

$P(Z = 1) = P((P_A = 1) \cdot P(P_B = 0)) = \frac{4}{9} \cdot \frac{5}{9}$

$P(Z = -1) = (P_A = 0, P_B = 1) = \frac{5}{9} \cdot \frac{4}{9}$

$P(Z = 0) = \frac{5}{9} \cdot \frac{5}{9} + \frac{4}{9} \cdot \frac{4}{9} = \frac{25}{81} + \frac{16}{81}$

$(P_A = 0, P_B = 0)$ OR $(P_A = 1 = P_B)$

$Var(Z) = E(Z^2) - (E(Z))^2$

$E(Z) = (-1) \cdot \frac{20}{81} + 0 \cdot \frac{41}{81} + (1) \cdot \frac{20}{81} = 0$

$E(Z^2) = (-1)^2 \cdot \frac{20}{81} + 0^2 \cdot (\frac{41}{81}) + (1)^2 (\frac{20}{81}) = \frac{40}{81}$

$Var(\det(M)) = \frac{40}{81}$

Q.12

$$\int_0^{100} e^x \{x\}dx = \int_0^1 e^x \{x\}dx + \underbrace{\int_1^2 e^x \{x\}dx}_{11} + \ldots$$

$\int_0^1 e^{x+1}\{x\}dx + \ldots$ (using shifting property!)

Q.13

$$A^{-1} = \begin{bmatrix} a & -b\sqrt{5} \\ b\sqrt{5} & a \end{bmatrix} / (a^2+5b^2)$$

$\Rightarrow \dfrac{a}{a^2+5b^2} \in \mathbb{Z}$ and $\dfrac{-b}{a^2+5b^2} \in \mathbb{Z}$. This occurs only at $\boxed{a=\pm 1, b=0}$

$$G = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\} = \{I, -I\}$$

So, If for all $A \in G \Rightarrow A^n \in G \; n \in \mathbb{N}$.

$$\sum_{B \in G} B = I - I = 0$$

[Since, $|a^2+5b^2| \geq |a|$ but we need $|a^2+5b^2| \leq |a|$ as integer (else fraction)]

Then $\boxed{b=0} \Rightarrow \dfrac{a}{a^2} \in \mathbb{Z} \Rightarrow 1/a \in \mathbb{Z} \Rightarrow \boxed{a=\pm 1}$

$$[\text{BONUS}: S = \left\{ \begin{bmatrix} a & 2c & 2b \\ b & a & 2c \\ c & b & a \end{bmatrix}; a, b, c \in \mathbb{Z} \right\}.$$

$G = \{A \in S : A^{-1} \in S\}\cdot$

Then

(A) $|M| = \pm 1$ for every $M \in G$

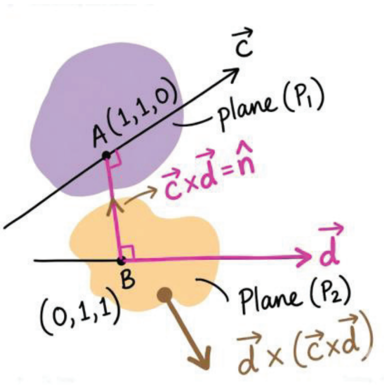
(B) $\exists M \in G$ such that $M \neq I \; M \neq -I$

(C) G has infinite element

(D) $\begin{bmatrix} 5 & 6 & 8 \\ 4 & 5 & 6 \\ 3 & 4 & 5 \end{bmatrix} \in S$

(Key: A, B, C, D !!! $\rightarrow$Check PLAYLIST ('MATRIX') FOR FULL SOLUTION)

Q.14



$A = L_1 \cap P_2$

$L_1 = (1+\lambda, 1+2\lambda, \lambda)$

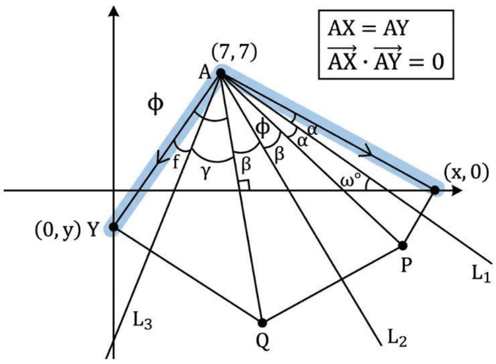
(Put in plane and get !)

(observe: AB is line of shortest distance)

$|AB| = \sqrt{2}$

Locus: M is mid point of $AB = (1/2, 1, 1/2)$
Since $M \in$ locus

Q.15



$$2(\alpha+\beta+\gamma)=\frac{\pi}{2}$$

$$\Rightarrow \boxed{\alpha+\beta+\gamma=\frac{\pi}{4}}$$

$$-((x-7)-7i)i=-7+(y-7)i$$

$$\Rightarrow -(x-7)i-7=-7+(y-7)i \Rightarrow \boxed{x+y=14}$$

$$\tan(\beta+\gamma)=\left|\frac{2+5}{1-10}\right|=\frac{7}{9}$$

$$\tan(\frac{\pi}{4}-\alpha)=\frac{7}{9} \Rightarrow \frac{1-\tan\alpha}{1+\tan\alpha}=\frac{7}{9}$$

$$\Rightarrow \frac{2}{-2\tan\alpha}=\frac{16}{-2} \Rightarrow \tan\alpha=\frac{1}{8}$$

$$\tan\alpha=\left|\frac{m+1}{1-m}\right|=\frac{1}{8}$$

$$\Rightarrow 8(m+1)=\pm(1-m)$$

$$8m+8=1-m$$

$$\boxed{\Rightarrow m=\frac{-7}{9}}$$

$$m=\frac{-7}{9}=\frac{-7}{x-7}$$

Gives $\boxed{x=16}$

$$8m+8=-1+m$$

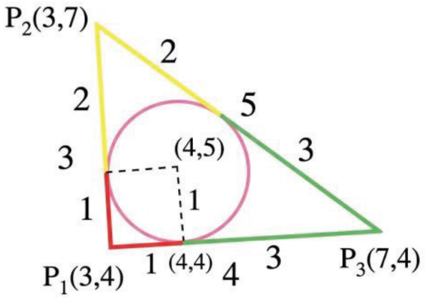
$$\Rightarrow \boxed{m=\frac{-9}{7}}$$

(Rejected! Think why?)

Q.16

n(S)=1

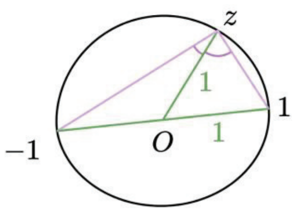
centre(4, 5) and r=1



M-2 use $\sqrt{S_1}$ = {Length of tangent and set up three equation:}

Q.17

$\frac{z-1}{z+1}=\lambda i, \quad \lambda>0 \Rightarrow |\mathbf{z}|=\mathbf{1}$



So, $z=e^{i\theta} \Rightarrow |z^3-z+2|^2=|z^3-z+2z\bar{z}|^2$

$=|z^2-1+2\bar{z}|^2=|\cos 2\theta+i\sin 2\theta-1+2\cos\theta-2i\sin\theta|^2$

$=(\cos 2\theta+2\cos\theta-1)^2+(\sin 2\theta-2\sin\theta)^2$

$=(2\cos^2\theta+2\cos\theta-2)^2+4\sin^2\theta(\cos\theta-1)^2$

$=4(\cos^2\theta+\cos\theta-1)^2+4\sin^2\theta(\cos\theta-1)^2$

$=4(t^2+t-1)^2+4(1-t^2)(t-1)^2$

$=16t^3-4t^2-16t+8, \quad t=\cos\theta$

$f'(t)=48t^2-8t-16=0$ (assume above as f(t))

$t=2/3 \text{ or } t=-1/2$

max occurs at $t=-1/2$ and $\boxed{\max=13}$

Q.18

RANK OF GANIT = 27 (n = 27)

$\sum_{n=1}^{120} P(w_n)=1 \Rightarrow \frac{1}{k}\sum\frac{n}{2^n}=1 \Rightarrow k=\sum_{n=1}^{120}\frac{n}{2^n}$

$\Rightarrow k=\frac{2^{120}-61}{2^{119}}$ (use, $k=\ldots$ and $\frac{k}{2}=\ldots$ and subtract!)

$\Rightarrow P(w_{27})=\frac{27\cdot 2^{92}}{2^{120}-61}$

$\alpha=92, \beta=120$

Q.19

A. $\phi(x)=f(f(f(g(x))))$ (Det $=12-10=2$)

B. $\phi(x)=f(g(h(h(x))))$ (where $f_1(x)=f(x), f_2(x)=g(x), f_3(x)=h(x)$)

(Determinant = 18)

C. $\phi(x)=h(h(g(f(x))))$ (Det $=18$)

Explanation: See how composition is matrix multiplication!

$h(x)=\dfrac{2x+1}{x+2}\cdot$ Define $M_n=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$

$g(x)=\dfrac{3x+1}{x+1}\cdot$ Define $M_g=\begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix}$

$f(x)=\dfrac{x}{x+1}\cdot$ Define $M_f=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$

$M_nM_nM_gM_f$ =h(h(g(f(x)))) **(simulation!)**

||

$=\begin{pmatrix}28 & 9\\ 26 & 9\end{pmatrix}$ decodes $=\dfrac{28x+9}{26x+9}$

Note: options must preserve determinant (Multiple of determinants!)

Det = 13 (which cannot be obtained by any combination of 1, 2, 3 .)

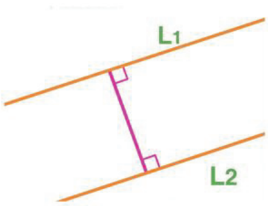
(Note: The Question takes hardly one-min if you know this marvellous idea!)

Q.20

$S:\vec{r}\cdot(6\hat{i}+8\hat{k})=105$

$T:\vec{r}\cdot(6\hat{i}+8\hat{k})=5$

planes are parallel:



$$d=\frac{|105-5|}{\sqrt{6^2+0^2+8^2}}=10$$

Q.21

$\{a, a+d, a+2d, \ldots, a+(n-1)d\}$ has same variance as $\{0, d, \ldots, (n-1)d\}$

$$\sigma^2 = \left(\frac{n^2-1}{12}\right)d^2$$

$$\sigma^2 \propto d^2$$

$$\Rightarrow \boxed{\sigma_2^2 < \sigma_3^2 < \sigma_1^2}$$

Q.22

$$P(x) = x^7(x-1) + x(x-1) + 15$$

$x \geq 1,\ P(x) \geq 0$

$x \leq 0,$

$P(x) \geq 0$

$x \in (0,1)$

$\Rightarrow P(x) > 0$

For $x > 0,\ h(x) = (a+x)^2 e^{(5-x)^2}$

$h'(0^+) = 2ae^{25}(1-5a)$

(Since $h(x)$ is even $\Rightarrow h'(0) = h'(0^+) = 0$ (Continuity))

$$\Rightarrow \boxed{a = 0, a = \frac{1}{5}} \Rightarrow s = \left\{0, \frac{1}{5}\right\}$$

Q.23

Put $x^2 = \sin\theta$ (Numerator)

$x^2 = \tan\theta$ (Denominator)

It evaluates to $(\sqrt{2})^2 = 2$.

Q.24

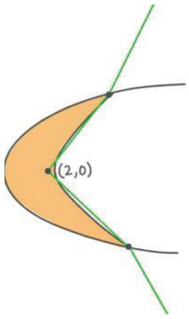
$y^2 = 4x$

$y = mx - 2m - m^3$

$\boxed{m^3 + (2-x)m + y = 0} \downarrow$

roots to be 3

$\boxed{27y^2 = 4(x-2)^3}$



Solving this with $y^2 = 4x$ gives:

$\boxed{x^3 - 6x^2 - 15x - 8 = 0} \Rightarrow x = 8$(factor theorem)

$$A = 2\left(\int_0^8 2\sqrt{x}\,dx - \int_2^8 \frac{2}{3\sqrt{3}}(x-2)^{3/2}\,dx\right)$$

$$\boxed{A = \frac{352\sqrt{2}}{15}}$$

Q.25

$$|M| = \begin{vmatrix} a+1 & 1 & 1 \\ 1 & b+1 & 1 \\ 1 & 1 & c+1 \end{vmatrix}$$

$= abc + ab + ac + bc$

$|M| \neq 0$

$\boxed{\begin{array}{l} (1,1,0),(1,0,1),(0,1,1) \rightarrow 3 \text{ vectors} \\ (1,1,1) \rightarrow 1 \text{ vector} \end{array}}$

$|M| = 0$ $\boxed{(0,0,0),(1,0,0),(0,1,0),(0,0,1)}$

$x + y + z = 0$ (∞-many) Two equations are same!

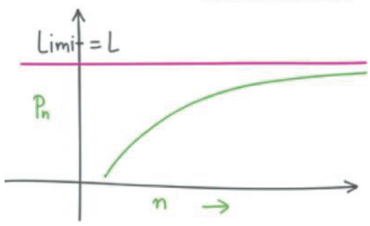
Q.26

$X_{n+1} \to X_n$ is even OR X_n is odd

$p_{n+1} = p_n(1-p) + (1-p_n)p \to (n+1)^{th}$ toss must be head

$\uparrow$

$(n+1)^{th}$ toss must be tail



$1 = (1-2p)L + p \Rightarrow \boxed{L = \frac{1}{2}}$

Q.27

$$S_{10}(x) = \sum_{r=1}^{10} \cos(rx) = \frac{\sin(5x)\cos\left(\frac{11x}{2}\right)}{\sin(x/2)}, x \in (0, \pi)$$

$$\sin(5x) = 0 \Rightarrow x = \frac{\pi}{5}, \frac{2\pi}{5}, \ldots$$

$$\cos\left(\frac{11x}{2}\right) = 0 \Rightarrow x = \frac{\pi}{11}, \frac{3\pi}{11}, \ldots$$

$$N^r : 2\sin(5x)\cos(5x) - \frac{2\sin^2(5x)\sin(x/2)}{\cos(x/2)}$$

$$= \frac{2\sin(5x)}{\cos(x/2)}\left(\cos(5x)\cos(x/2) - \sin(5x)\sin(x/2)\right)$$

$$= \frac{2\sin(5x)\cos\left(\frac{11x}{2}\right)}{\cos(x/2)}$$

$$\lim_{x\to x_0} = \lim_{x\to x_0} \frac{2\sin(5x)\ \cos\left(\frac{11x}{2}\right)}{\cos(x/2)\ \cos\left(\frac{11x}{2}\right)} = \frac{2\sin\left(\frac{5\pi}{11}\right)}{\cos\left(\frac{\pi}{22}\right)} = \boxed{2}$$

Q.28

$$\text{let } u = x - 1/2 \Rightarrow du = dx$$

$$x \in [0,1] \xrightarrow{\text{map}} u \in [-1/2, 1/2]$$

$f_1(u) = \cos(\pi u) \to$ even

$f_2(u) = 2u\cos(\pi u) \to$ odd

$f_{n+2}(u) = 2uf_{n+1}(u) - f_n(u)$ (in u-domain)

For $n = 1, f_1(u) = \text{even} \Rightarrow \boxed{I_1 = \frac{2}{\pi}}$

$n = 2, f_2(u) = \text{odd} \Rightarrow \boxed{I_2 = 0}$

So, $\boxed{I_{2k} = 0 \text{ for all } k \in \mathbb{N}}$

$$I_3 = \int_{-1/2}^{1/2} (4u^2\cos(\pi u) - \cos(\pi u))du = \frac{2}{\pi} - \frac{16}{\pi^3}$$

Since each subsequent sequence increases in polynomial degree so, $I_{odd} \to$ distinct.

Q.29

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$$

Since, if $\vec{u} = \vec{v} = \vec{w} = \vec{a} \Rightarrow \vec{a} \times (\vec{a} \times \vec{a}) = \vec{0} \in S$

Take extreme case $|\vec{a}| = |\vec{b}| = 1$

$S = \{0, \vec{a}, -\vec{a}, \vec{b}, -\vec{b}\}$

What if for $\vec{u}, \vec{v} \in S$ (in $\mathbb{R}^3$ set of all vectors), $\vec{u} \times \vec{v} \in S$.

Then minimum element is

$\{\vec{0}\}$ or $\{\vec{0}, \vec{a}, -\vec{a}, \vec{b}, -\vec{b}, \vec{c}, -\vec{c}\}$ orthogonal unit vector!

Q.30

$$F(k) = \int_a^b |f(x) - k|\,dx$$

$$h(\alpha) = \int_a^\alpha (f(\alpha) - f(x))dx + \int_\alpha^b (f(x) - f(\alpha))dx$$

$$\Rightarrow \frac{dh}{d\alpha} = \int_a^\alpha \frac{df}{d\alpha}dx + \int_\alpha^b -\left(\frac{df}{d\alpha}\right)dx = 0 \text{ (for minimum)}$$

Therefore, $\alpha = \frac{a+b}{2}$ (as f is monotonic)

The given function is even so, $I = 2\int_0^8 |f(x) - k|\,dx$

Since f is increasing in

So $k = f\left(\frac{0+8}{2}\right) = f(4) = 0$

$$I_{min} = 2\int_0^8 |f(x)|\,dx = \boxed{8\pi + \frac{256}{3}}$$

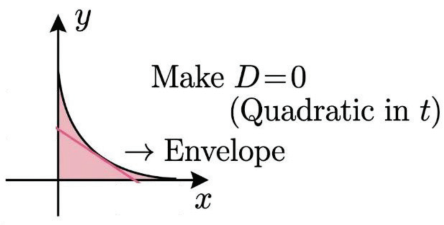
Q.31

$$X = \begin{bmatrix}1\\3\\6\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$M^n\begin{bmatrix}1\\3\\6\end{bmatrix} = \begin{bmatrix}1\\3\\6\end{bmatrix} \forall n \in \mathbb{N}.$$

$$M^n\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{4^n}\\0\\0\end{bmatrix} \forall n \in \mathbb{N}.$$

Q.32



$$\boxed{\sqrt{x} + \sqrt{y} = 1}$$

Q.33

$$T(A,B)=\binom{A-1}{B-1}(\text{star}-\text{bar})$$

$$\sum_{i=3}^{12}T(i+1,3)=\sum_{i=3}^{12}\binom{i}{2}\;=\binom{3}{2}+\binom{4}{2}+\ldots+\binom{12}{2}$$

$$=\binom{3}{3}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{12}{2}-1$$

$$=\binom{13}{3}-1$$

$$\boxed{\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}}$$

Q.34

$$a=-(r_1+r_2+r_3+r_4)$$
$$b=r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4$$
$$c=-(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)$$
$$b-a-c=(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)+(r_1+r_2+r_3+r_4)+(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)$$

ByA.M $-$ G.M.

$$b-a-c\geq 14((r_1r_2r_3r_4)^7)^{\frac{1}{14}}$$
$$=14\sqrt{d}$$
$$(b-a-c)^2\geq 196d$$

Q.35

$$\lambda^3-\text{tr}(C)\lambda^2+\frac{1}{2}((\text{tr }C)^2-\text{tr}(C^2))\lambda-\det(C)=0$$

$$\Rightarrow \lambda^3-\frac{1}{2}\text{tr}(C^2)\lambda-\det(C)=0$$

$$\text{Cayley}-\text{Hamilton}\rightarrow C^3-\frac{C}{2}\text{tr}(C^2)-\det(C)=0$$

Take trace on both sides:

$$\text{tr}(C^3)-\frac{\text{tr}(C)}{2}\text{tr}(C^2)-\det(C)\text{tr}(I)=0$$

$$\Rightarrow \text{tr}(C^3)-\det(C)\times 3=0$$

$$\det(AB-BA)=\frac{\text{tr}((AB-BA)^3)}{3}$$

HOLDS ALWAYS

Q.36

$f(2x)-f(x)=x$ for all $x\in\mathbb{R}$.

$$f(x)-f\left(\frac{x}{2}\right)=\frac{x}{2}$$

$$f(x/2)-f\left(\frac{x}{4}\right)=\frac{x}{4}$$

$$\vdots$$

$$f\left(\frac{x}{2^{n-1}}\right)-f\left(\frac{x}{2^n}\right)=\frac{x}{2^n}$$

Summing up:

$$f(x)-f\left(\frac{x}{2^n}\right)=x\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}\right)$$

When $n\to\infty$, since f is continuous

$\Rightarrow f(x)-f(0)=x$

$\Rightarrow \boxed{f(x)=x+1}$

(Continuity is CRUCIAL!)

Q.37

$f(k)(1+\ln f(k))=k$

$$\Rightarrow \frac{f(k)}{k}=\frac{1}{1+\ln f(k)}$$

$$\lim_{k\to\infty}\frac{f(k)\ln k}{k}=\frac{\ln k}{1+\ln f(k)}=\frac{\ln f(k)+\ln(1+\ln f(k))}{1+\ln f(k)}$$

let $f(k) = t$

$$\lim_{t\to\infty}\frac{\ln t+\ln(1+\ln t)}{1+\ln t}$$

$$=\lim_{t\to\infty}\left(\frac{1+\dfrac{\ln(1+\ln t)}{\ln t}}{\dfrac{1}{\ln t}+1}\right)$$

=1

Q.38

$$\lim_{n\to\infty}\left((8n^3+4n^2+n+11)^{1/3}-2n\right)=\frac{1}{3}$$

and $\lim_{n\to\infty}\left((4n^2+n+9)^{1/2}-2n\right)=\dfrac{1}{4}$

so, $L=\dfrac{1}{3}-\dfrac{1}{4}=\dfrac{1}{12}$

Q.39

$$A_1^n = \begin{bmatrix} 1 & 2n-1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = A_{2n-1}$$

$$A_k^n = \begin{bmatrix} 1 & (nk+n-1) & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A_{nk+n-1}$$

$$A_k^n = A_{75} \Rightarrow nk + n - 1 = 75 \Rightarrow n(k+1) = 76$$

$\therefore 76 = 2^2 \cdot 19$, so we have only five possibility:

$(k,n) \in \{(1,38),(3,19),(18,4),(37,2),(75,1)\}$

Note : $|A+B| \neq |A| + |B|$ ingeneral

In option D R.H.S $\det \neq 0$, But L.H.S $\det = 0$

Q.40

$$y_2 - x\cot\frac{\theta}{2}$$

$$\int_{-\cot\frac{\theta}{2}}^{0} -x\cot\frac{\theta}{2} - x^2 = \frac{9}{2}$$

$$\frac{\cot^3\frac{\theta}{2}}{6} = \frac{9}{2}$$

$$\cot\frac{\theta}{2} = 3$$

$$\cos\left(\frac{\theta}{2}\right) = \frac{3}{\sqrt{10}},\ \sin\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{10}}$$

$$\frac{\cos\theta}{1-\sin\theta} = \frac{\frac{18}{10} - \frac{1}{6}}{1 - \frac{6}{10}} = 2$$

Q.41

$$\alpha_n = (2+\sqrt{3})^n, \beta_n = (2-\sqrt{3})^n, \beta_n \in (0,1)$$
$$[\alpha_n] = \alpha_n + \beta_n - 1 \Rightarrow f_n = \alpha_n - [\alpha_n] = 1 - \beta_n$$
$$\lim_{n\to\infty} f_n = 1$$

MULTI CORRECT

Q.42

Ans. ABCD

Gives $\lambda = \pm 4$ ($\lambda = \pm 4$ rejected as same line)

So. S={-4} $\Rightarrow w = -4$

$$P = \begin{bmatrix} 1 & 2 & -4 \\ 2 & 1 & 3 \\ 1 & 4 & -4 \end{bmatrix} \Rightarrow |P| = 27$$

$$P = \frac{K}{|P|}\operatorname{adj}(P) \rightarrow k = 27$$

$$PQ = kI \Rightarrow Q = \frac{k}{|P|}\operatorname{adj}(P)\left(\text{as } K = |P| = 27\right)$$

$|Q| = 3^6$

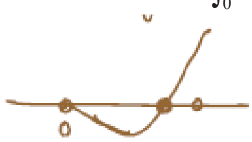
$\Rightarrow |\operatorname{adj}(Q)| = (3^6)^2 = 3^{12}$

Q.43

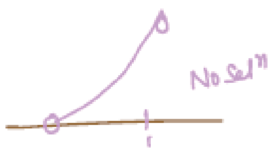
(A) $g(x) = \sin(\pi x)\int_0^x f(t)dt$ (ApplyRolle's)

(B) $h(x) = \int_0^x f(t)dt - x^{2026}$

(C) $p(x) = x^{2026} - \int_0^x f(t)\cos t\,dt$



(D) $F(x) = e^x\int_0^x f(t)dt - 1 \Rightarrow$



$f' = e^x - f(t) > 0$

Q.44

$\vec{v}\times(\vec{u}\times\vec{v})=\vec{u}\Rightarrow|\vec{v}|^2\ \vec{u}-(\vec{u}\cdot\vec{v})\vec{v}=\vec{u}$

$\Rightarrow|\vec{v}|^2=1$

Since, $|\vec{u}||\vec{v}|\sin 90^\circ=|\vec{w}|=\sqrt{6}\Rightarrow|\vec{u}|=\sqrt{6}$

$\alpha^2+\beta^2+\gamma^2=6$

So, $[\vec{u}\vec{v}\vec{w}]=|\vec{u}|^2=6$

Since, $\vec{u}\cdot\vec{w}=0\Rightarrow\alpha+\beta=2\gamma\Rightarrow\alpha+\beta+\gamma=3\gamma$

Using $\dfrac{(\alpha+\beta)^2}{2}\le\alpha^2+\beta^2\Rightarrow\dfrac{(2\gamma)^2}{2}+\gamma^2\le\alpha^2+\beta^2+\gamma^2$

$\Rightarrow\gamma_{max}=\sqrt{2}\Rightarrow|\alpha+\beta+\gamma|_{max}=3\sqrt{2}$

Q45.

$f'(2n+2)^-=a_n+\dfrac{b_n}{\pi}$

$f'(2n+2)^+=4a_{n+1}(n+1)$

$\Rightarrow a_n+\dfrac{b_n}{\pi}=4a_{n+1}(n+1)$ for all $n\in\mathbb{Z}$

For this to be true for all $n\in\mathbb{Z}$, $a_n=b_n=0$

Q.46

$$T_k = \tan^{-1}\left(\frac{2}{8k^2-4k-1}\right)$$

$$= \tan^{-1}\left(\frac{10}{40k^2-20k-5}\right)$$

$$= \tan^{-1}\left(\frac{\frac{6k+1}{2(k+1)}-\frac{6k-5}{2k}}{1+\frac{(6k+1)(6k-5)}{4k(k+1)}}\right)$$

$$= \tan^{-1}\left(\frac{6k+1}{2k+2}\right) - \tan^{-1}\left(\frac{6k-5}{2k}\right)$$

$$S = \lim_{n\to\infty}\left(\tan^{-1}\left(\frac{6n+1}{2n+2}\right) - \tan^{-1}\frac{1}{2}\right) = \tan^{-1}3 - \tan^{-1}\frac{1}{2} = \tan^{-1}1 = \frac{\pi}{4}$$

EXTRA QUESTION:

Let $\theta, \phi \in [0, 2\pi]$ be such that the system of linear equations in x and y

$$\begin{vmatrix} \cos\theta & \sin\phi \\ \sin\theta & \cos\phi \end{vmatrix} x + \begin{vmatrix} \cos\phi & \sin\theta \\ \sin\phi & \cos\theta \end{vmatrix} y = 0$$

$\begin{vmatrix} \cos\theta & 1 \\ \cos\phi & 1 \end{vmatrix} x + \begin{vmatrix} \sin\theta & \sin\phi \\ 1 & 1 \end{vmatrix} y = 0$ has a non-trivial solution and the angles θ, ϕ also satisfies $\dfrac{\cos\theta - \cos\phi}{\sin\theta - \sin\phi} < 1$ (given $\sin\theta \neq \sin\phi$). Then θ satisfy:

(A) $\theta \in (0, \frac{\pi}{4})$ (B) $\theta \in (\frac{3\pi}{4}, \pi)$

(C) $\theta \in (\frac{5\pi}{4}, \frac{3\pi}{2})$ (D) $\theta \in (\frac{7\pi}{4}, 2\pi)$

Sol:

For non-trivial solution

$\cos(\theta+\phi)((\sin\theta - \sin\phi) - (\cos\theta - \cos\phi)) = 0$

So, $\cos(\theta+\phi) = 0$

Or $\sin\theta - \cos\theta = \sin\phi - \cos\phi$

The inequality $\tan\left(\dfrac{\theta+\phi}{2}\right) > -1$..

Let $Y = \dfrac{\theta+\phi}{2}$, Since $\theta, \phi \in [0, 2\pi]$ so, $Y \in [0, 2\pi]$.

$\tan Y > -1$ holds unless Y falls in $[\frac{3\pi}{4}, \pi) \cup [\frac{7\pi}{4}, 2\pi]$

So, $\theta + \phi$ must avoid $[\frac{3\pi}{2}, 2\pi) \cup [\frac{7\pi}{2}, 4\pi]$

Q.47

$$J_n = \sum_{k=1}^{n} k\int_0^{\alpha} \sin\left(\frac{k\pi x}{\alpha}\right)\sin\left(\frac{\pi x}{\alpha}\right)dx$$

Let $u = \frac{\pi x}{\alpha}, dx = \frac{\alpha}{\pi}du$

$$L_k = \frac{\alpha}{\pi}\int_0^{\pi} \sin(ku)\sin(u)du$$

Using $\int_0^{\pi} \sin(ku)\sin(u)du = \begin{cases} 0, & k \neq 1 \\ \pi/2 & \text{if } k = 1 \end{cases}$

For $k = 1, L_1 = \alpha/2 \; k > 1, L_k = 0$

$$J_n = L_1 + L_2 + \cdots + L_n = \alpha/2 + 0 + 0 + \cdots = \frac{\alpha}{2}$$

Q.48

$$\boxed{\begin{aligned} &P(A \cap B) = P(A)P(B) \\ &P(A \cap C) = P(A)P(C) \\ &P(B \cap C) = P(B)P(C) \\ &P(A \cap B \cap C) = P(A)P(B)P(C) \end{aligned}}$$

All needed (last fails here!)

Q.49

$$|\vec{a}|^2 = b^2c^2\sin^2\theta + 4b^2 = 16(1+\sin^2\theta) \geq 16$$

$$\text{but} |\vec{a}| \leq 4 \Rightarrow \sin\theta = 0 \Rightarrow \theta \in \{0, \pi\}$$

$$|2\vec{b}\times\vec{c} + 5\vec{b} + \vec{c}|^2 = 4b^2c^2\sin^2\theta + 25b^2 + c^2 + 10bc\cos\theta$$

$$= 64\sin^2\theta + 104 + 40\cos\theta$$

$$\boxed{\begin{aligned} &\theta = 0 \rightarrow 144 \\ &\theta = \pi \rightarrow 64 \end{aligned}}$$

So, $|2\vec{a} + \vec{b} + \vec{c}| \in \{8, 12\}$

Q.50

(A) $|PAP^{-1}| = |P||A||P^{-1}| = |A|$

$Tr(PAP^{-1}) = Tr(AP^{-1}P) = Tr(AI) = Tr(A)$ (Trace is Commutative)

Existence can be shown by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and multiply.

(B) $A^2 = \begin{bmatrix} k^2 & k + \frac{1}{k} \\ 0 & \frac{1}{k^2} \end{bmatrix}$ $A^3 = \begin{bmatrix} k^3 & k^2 + 1 + \frac{1}{k^2} \\ 0 & \frac{1}{k^3} \end{bmatrix}$

b_n follows G.P. $\Rightarrow b_n = k^{n-1} + k^{n-3} + \ldots + k^{-(n-1)}$

$\Rightarrow \boxed{b_n = \frac{k^n - k^{-n}}{k - k^{-1}} = \frac{k^{n+1} - k^{1-n}}{k^2 - 1}}$

(C) $|A(k)| = 1$, eigen value of A^{10} are $\{2^{10}, 2^{-10}\}$

(Reason : $A^{-1} = \frac{adj(A)}{|A|}$, here $|A| = 1$)

(D) $A(k)^{-1} = \frac{adjA(k)}{|A(k)|} = adjA(k)$

$|adjA(k)| = |A(k)|^2 = 1$

So, $|2\ adj\ A(k)| = 4\ |adj\ A(k)| = 4$ (independent of k)

Q.51

(A) $I = \int_{1/4}^{3/4} f(f(1-x))dx = \int_{1/4}^{3/4} f(1-f(x))dx$

$= \int_{1/4}^{3/4} (1-f(f(x)))dx$

$= \frac{1}{2} - I$

$\Rightarrow I = \frac{1}{4}$

(B) $\int_0^1 xf''(x)dx = xf'(x)\Big|_0^1 - \int_0^1 f'(x)dx$

$= f'(1) - (f(1) - f(0))$

$= f'(1) - \frac{1}{3} = f'(0) - \frac{1}{3}$ $\boxed{f'(x) = f'(1-x)}$

$$\boxed{\begin{array}{l} f'(x) = \dfrac{2 \cdot 4^x \ln 4}{(4^x+2)^2} \\ \Rightarrow f'(0) = \dfrac{2\ln 4}{9} = \dfrac{4\ln 2}{9} \end{array}}$$

(C) $f'(x) = \frac{2 \cdot 4^x \ln 4}{(2+4^x)^2}$

$f(x) = \frac{4^x}{2+4^x}$

$\Rightarrow T_r = \frac{2}{4^r}$

$\Rightarrow \sum_{r=1}^{\infty} T_r = 2\sum_{r=1}^{\infty} 4^{-r} = 2 \times \frac{1}{1-1/4} = \frac{8}{3}$

Q.52

If $t \geq 0, L = 0 \Rightarrow t < 0$. Let $\frac{1}{n} = x$

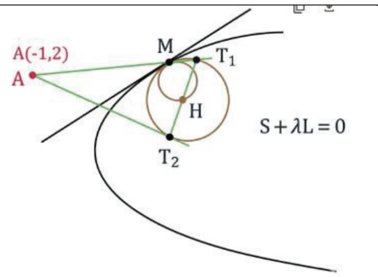
$L = \lim_{x\to 0} \frac{e(1-x)^{1/x} - 1}{x^{-t}}$ (L-Hospital's)

$= \lim_{x\to 0} \frac{e(1-x)^{1/x}\left(\frac{1}{x^2-x} - \frac{\ln(1-x)}{x^2}\right)}{-\alpha x^{-\alpha-1}}$

N^r is $\boxed{-\frac{1}{2}}$

D^r must be such that $\boxed{\alpha = -1}$

Q.53



$S_\lambda : (x-1)^2+(y-2)^2+\lambda(x-y+1)=0$

$S_\lambda : x^2+y^2+x(\lambda-2)+y(-4-\lambda)+(\lambda+5)=0$

T=0 gives: (α,β) be midpoint

$\boxed{(\lambda-4)x-\lambda y+(4-\lambda)=0}$

T=S_1 gives: (α,β) be midpoint

$x(2\alpha+\lambda-2)+y(2\beta-\lambda-4)=2\alpha^2+2\beta^2+\alpha\lambda-2\alpha-\beta\lambda-4\beta$

Compare: $\boxed{\dfrac{2\alpha+\lambda-2}{\lambda-4}=\dfrac{2\beta-\lambda-4}{-\lambda}}$

$\boxed{\Rightarrow \lambda=\dfrac{4(\beta-2)}{\alpha+\beta-1}}$

Since (α,β) lies on chord,

So, $\alpha(\lambda-4)-\lambda\beta+(4-\lambda)=0$

$\Rightarrow (\beta-2)(\alpha-\beta-1)=(\alpha-1)(\alpha+\beta-1)$

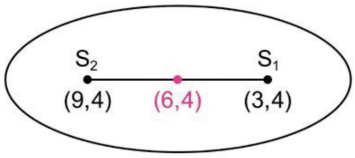
$\Rightarrow \boxed{\alpha^2+\beta^2-2\beta-1=0}$

$\Rightarrow \boxed{\alpha^2+(\beta-1)^2=2}$

$\boxed{h^2+(k-1)^2=2}$

Note: what if "NOT NECESSARILY DISTINCT" is taken away?)

Q.54



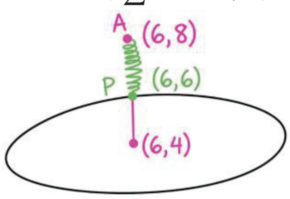
$$\frac{(x-6)^2}{25}+\frac{(y-4)^2}{16}=1 \ \rightarrow \text{Ellipse}$$

$$e=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$$

$$\sum_{k=1}^{n} x_k=\sum_{k=1}^{n}(6+5\cos\frac{2\pi k}{n})=6n+\boxed{0} \ \text{roots of unity}$$

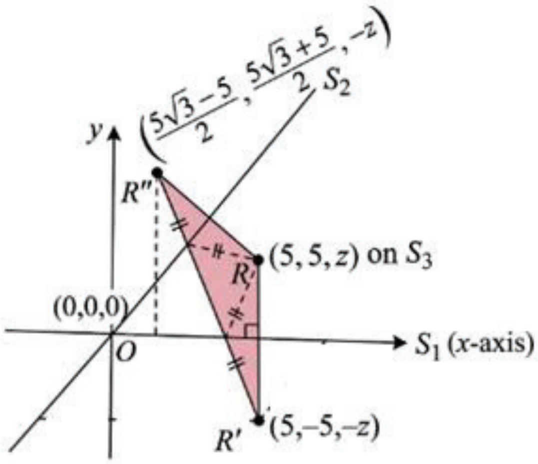
$$\sum_{k=1}^{n} x_k y_k=\sum_{k=1}^{n}(24+24\cos\theta_k+20\sin\theta_k+20\sin\theta_k\cos\theta_k)$$

$$=24n+\underbrace{0+0+0}_{(\uparrow\sum \text{roots of unity}=0)}$$



$$\Rightarrow |AP|=2$$

Q.55



Minimum will be achieved when R lies on $R'R''$.

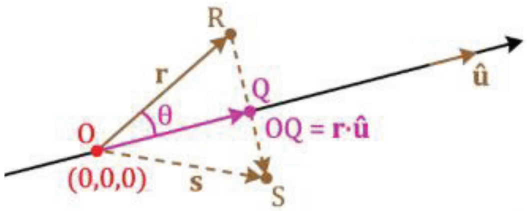
$$S(z)=\sqrt{\left(\frac{5\sqrt{3}-15}{2}\right)^2+\left(\frac{5\sqrt{3}+15}{2}\right)^2+(z-(-z))^2}$$

$$=\sqrt{150+4z^2}$$

$$=\sqrt{150}\,(\text{at } z=0)$$

$$=5\sqrt{6}$$

Reflection principle:



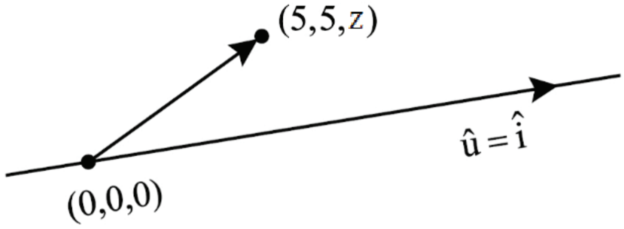
$$OQ = r\cos\theta = \frac{\vec{r}\cdot\hat{u}}{|\hat{u}|} = (\vec{r}\cdot\hat{u})$$

$$\overrightarrow{OQ} = (\vec{r}\cdot\hat{u})\hat{u}$$

$$\vec{r}+\vec{s} = 2(\vec{r}\cdot\hat{u})\hat{u}$$

$$\Rightarrow \boxed{\vec{s} = 2(\vec{r}\cdot\hat{u})\hat{u}-\vec{r}}$$

<u>So in above:</u>



$$\vec{s} = 2((5,5,z)\cdot\hat{i})\hat{i}-(5,5,z)$$

$$\vec{s} = 2\times 5\hat{i}-(5,5,z)$$

$$\vec{s} = (5,-5,-z)$$

$$\vec{s} = 2\left((5,5,z)\cdot\left(\frac{\hat{i}}{2}+\frac{\sqrt{3}}{2}\hat{j}\right)\right)\left(\frac{\hat{i}}{2}+\frac{\sqrt{3}}{2}\hat{j}\right)-(5,5,z)$$

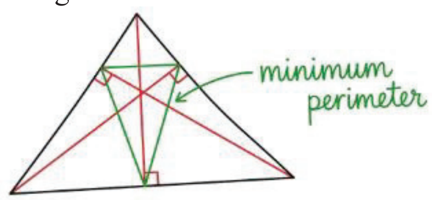
$$= 2\left(\frac{5}{2}+\frac{5\sqrt{3}}{2}\right)\left(\frac{\hat{i}}{2}+\frac{\sqrt{3}}{2}\hat{j}\right)-(5,5,z)$$

$$= \left(\frac{5+5\sqrt{3}}{2}\right)(\hat{i}+\sqrt{3}\hat{j})-(5,5,z)$$

$$= \left(\frac{5+5\sqrt{3}}{2}\right)\hat{i}+\frac{5\sqrt{3}}{2}\hat{j}+\frac{15}{2}\hat{j}-5\hat{i}-5\hat{j}-z\hat{k}$$

$$= \left(\frac{5\sqrt{3}-5}{2}\right)\hat{i}+\left(\frac{5\sqrt{3}+5}{2}\right)\hat{j}-z\hat{k}$$

<u>Note:</u> for 2-D acute angled triangle case minimum perimeter occurs at orthoc triangle.



Q.56

$$S_\alpha = 1+\alpha+\ldots+\alpha^m = \frac{\alpha^{m+1}-1}{\alpha-1} = \alpha^{m+2}-\alpha$$

Since, $\alpha^2 = \alpha+1 \Rightarrow \frac{1}{\alpha} = \alpha-1$

$\Rightarrow T_m = \alpha^{m+2}+\beta^{m+2}-(\alpha+\beta) = \alpha^{m+2}+\beta^{m+2}-1$

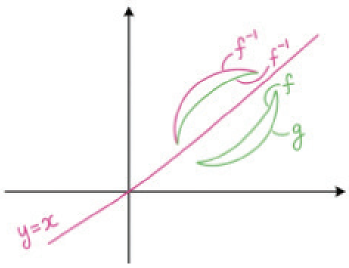
let $L_k = \alpha^k+\beta^k$

So, $L_k = L_{k-1}+L_{k-2} \quad (k \geq 3)$

L_k : {1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322}

$\boxed{T_m = L_{m+2}-1\left(\text{So, take 1 from } L_k \text{ to get } T_m\right)}$

Q.57



Since , $x \geq x^2$

$x^2 \geq x^3$ in $[0,1]$

$$A_2 = \int_0^1 (f-g)dx = \frac{1}{8} = \text{Area } (f^{-1}, g^{-1}) = A_1$$

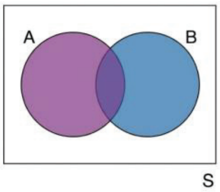
$$A_3 = 2\int_0^1 |x-f(x)|dx = \frac{1}{6}$$

$$A_4 = 2\int_0^1 |x-g(x)|dx = \frac{5}{12}$$

(Tests Graphical skill and function behaviour!)

Q.58

Note: $P(A)+P(B)>1(\text{So}, P(A\cap B)\neq 0)$



$$1\geq P(A\cup B)\geq \max(P(A),P(B))\geq \frac{2}{3}$$

$$P(A\cap B)=P(A)+P(B)-P(A\cup B)$$

$$=\frac{19}{15}-P(A\cup B)$$

$$P(A\cap B)_{\min}=\frac{4}{15}(\text{at } P(A\cup B)=1)$$

$$P(A\cap B)_{\max}=\frac{3}{5}(\text{at } P(A\cup B)=\frac{2}{3})$$

$$P(A\cap \bar{B})=P(A)-P(A\cap B)=\frac{3}{5}-P(A\cap B)\leq \frac{3}{5}-\frac{4}{15}\leq \frac{1}{3}$$

$$P(\bar{A}\,|\,B)=\frac{P(\bar{A}\cap B)}{P(B)}=\frac{P(B)-P(A\cap B)}{P(B)}$$

$$=\left(1-\frac{3}{2}P(A\cap B)\right)\in\left[\frac{1}{10},\frac{3}{5}\right]$$

Q.59

$$\sin x\approx x-\frac{x^3}{6},\ \cos x\approx 1-\frac{x^2}{2}$$

$$N^r=(a-1-b)x+\left(\frac{b}{2}-\frac{a}{6}\right)x^3.$$

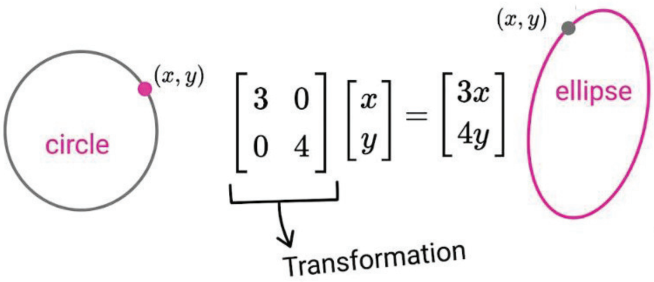
For L to exist, a - b = 1 and $\frac{b}{2}-\frac{a}{6}=1.$

$\Rightarrow a=9/2, b=7.$

Q.60

(1) Set A: $\{\cos\theta_K + i\sin\theta_K\}$

Set B: $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$



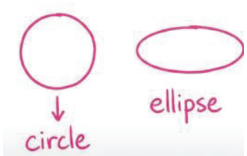
$e = \sqrt{1 - \dfrac{9}{16}} = \dfrac{\sqrt{7}}{4}$ $S_F = \{\sqrt{7}i, -i\sqrt{7}\}$

(2) $\sum\limits_{w\in B} |w|^2 = \sum\limits_{w\in B} (9\cos^2\theta_k + 16\sin^2\theta_k) = 125.$ as ($\sum \cos 2\theta_k = 0$ for a regular polygon.)

(3) $d(w, F_1) = b - ey = 4 - 4e\sin\theta_k$

$\sum \sin\theta_k = 0 \Rightarrow \sum d(w, F_1) = 40.$

(4) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 4y \end{bmatrix}.$

Transformation $\Rightarrow$ 

$\begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix} = \dfrac{\text{Area (ellipse)}}{\text{Area (circle)}} = \boxed{12}$

Q.61

$\vec{\omega}\cdot\hat{u} + (\vec{\omega}\times\hat{u})\cdot\hat{u} = \hat{u}\cdot\hat{v}.$

$\Rightarrow \vec{\omega}\cdot\hat{u} = \cos\theta.$

$|\vec{\omega}+\vec{\omega}\times\hat{u}|^2 = |\hat{v}|^2$

$\Rightarrow |\vec{\omega}|^2 + |\vec{\omega}\times\hat{u}|^2 + \underbrace{2\vec{\omega}\cdot(\vec{\omega}\times\hat{u})}_{=0} = 1$

$\Rightarrow |\vec{\omega}|^2 + (|\vec{\omega}|^2 - \cos^2\theta) = 1$

$$\Rightarrow |\omega| = \sqrt{\frac{1+\cos^2\theta}{2}}$$

(A) $V = |[\omega\hat{u}\hat{v}]| = [\vec{\omega}\cdot(\hat{u}\times\hat{v})].$

$V = [\vec{\omega}\hat{u}(\vec{\omega}+\vec{\omega}\times\hat{u})]$

$= [\vec{\omega}\,\hat{u}\,\vec{\omega}] + [\vec{\omega}\ \hat{u}\ \vec{\omega}\times\hat{u}]$

$$= |\vec{\omega}\times\hat{u}|^2 = \frac{1+\cos^2\theta}{2} - \cos^2\theta$$

$$= \frac{\sin^2\theta}{2}, \theta\in[0,\pi].$$

$\boxed{V_{max} = 1/2}$

(B) The equation can be written as $(I-K)\vec{w} = \hat{v}$ where K is a skew symmetric matrix representing the cross product with $\hat{u}$. The determinant of (I + K) is $(1+|\hat{u}|^2) = 2 \neq 0$, so, $\vec{w}$ has unique value = $(I+K)^{-1}\hat{v}$ (Inverse is unique).

So, Number of $\vec{w}$ is 1

(C) $|\vec{w}| = \sqrt{\frac{1+\cos^2\theta}{2}} \in \left[\frac{1}{\sqrt{2}}, 1\right]$

(D) $\vec{w} + \vec{w}\times\hat{u} = \hat{v}$

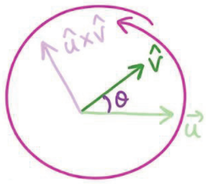
$\Rightarrow (\vec{w}\times\hat{u}) + (\vec{w}\times\hat{u})\times\hat{u} = \hat{v}\times\hat{u}$

$\Rightarrow (\vec{w}\times\hat{u}) + (\cos\theta\hat{u} - \vec{w}) = \hat{v}\times\hat{u}$

$\Rightarrow \vec{w} + (\vec{w} - \cos\theta\hat{u} + \hat{v} \times \hat{u}) = \hat{v}$

$((\vec{w} \times \hat{u}) \times \hat{u} = \cos\theta\hat{u} - \vec{w})$

$\boxed{\vec{w} = \frac{1}{2}[\hat{v} + (\hat{u} \times \hat{v}) + (\hat{u}\cdot\hat{v})\hat{u}]}$ $\Rightarrow$ You can solve all problems using this



unique $\vec{\omega}$ for a given $\hat{u}$ and $\hat{v}$

$\vec{\omega}$ is not always coplanar with $\hat{u}$ and $\hat{u} \times \hat{v}$.

In fact $\vec{\omega} = t\hat{u} + r\hat{v} + s(\hat{u} \times \hat{v})$ $\rightarrow$ Basic vector

Remember: $[\vec{a}\vec{b}\vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$

$[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}] = [\vec{a}\vec{b}\vec{c}]^2$

<u>Matrix Logic Explanation:</u>

Let K be the skew-symmetric matrix such that $K\vec{x} = \hat{u} \times \vec{x}$ (column vector).

So, $\vec{\omega} + \vec{\omega} \times \hat{u} = \hat{v} \Rightarrow (I - K)\vec{\omega} = \hat{v}$

$(I + K)(I - K)\vec{\omega} = (I + K)\hat{v}$

$\Rightarrow (I - K^2)\vec{\omega} = \hat{v} + \hat{u} \times \hat{v}$

$K^2\vec{\omega} = \hat{u} \times (\hat{u} \times \vec{\omega}) = (\cos\theta)\hat{u} - \vec{\omega}$

$\vec{\omega} - ((\cos\theta)\hat{u} - \vec{\omega}) = \hat{v} + \hat{u} \times \hat{v}$

$\Rightarrow \boxed{\vec{\omega} = \frac{1}{2}(\hat{v} + \hat{u} \times \hat{v} + (\cos\theta)\hat{u})}$

Q.62

$MB \equiv \vec{a} \times \vec{b}$ (as M is Skew Symmetric! write and feel)

$X = M^2B \equiv \vec{a} \times (\vec{a} \times \vec{b})$

$X = (\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b}$

$X = (A^T B)A - kB$

(A) $A^T X \equiv \vec{a} \cdot (\vec{a} \times (\vec{a} \times \vec{b})) = 0$

(as $\vec{a} \times \vec{v} \perp \vec{a}$)

(B) $M^3 = -kM$ (crossproduct)

$\Rightarrow M^4 = -kM^2$

$\Rightarrow M^{2n} = (-k)^{1012} M^2$

$\Rightarrow \boxed{\mathbf{M^{2026}B = (A^T A)^{1012} X}}$

as $X = M^2B$

(C) $X^T MB \equiv X \cdot (\vec{a} \times \vec{b}) = 0$

since X is linear combination of $\vec{a}$ and $\vec{b}$ $[\vec{X}\ \vec{a}\ \vec{b}] = 0$

(D) $A^T B \equiv \vec{a} \cdot \vec{b} = 0$

$\Rightarrow X = (A^T B)A - kB = -kB$

$\boxed{\Rightarrow \mathbf{M^2B = -kB}}$

So B is an eigenvector of M^2 with eigenvalue

$\lambda = -A^T A \equiv -\vec{a} \cdot \vec{a}$.

<u>Alternate:</u>

$$M^2 = AA^T - (A^T A)I \text{ for any} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{and } M = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Since $M^2B \equiv \vec{a} \times (\vec{a} \times$ 😄)

(A) $A^T X = A^T(M^2B) = (A^T M)(MB) = 0$

Since MA = 0 (see multiplication)

$\Rightarrow A^T M^T = 0 \Rightarrow -A^T M = 0 \Rightarrow A^T M = 0$

(B) $M^{2026}B=(A^TA)^{1012}X$

$M^3=-(A^TA)M$

.

.

.

$M^{2026}=k^{1012}M^2$

$\Rightarrow \boxed{\mathbf{M^{2026}B=(A^TA)^{1012}X}}$

$as X=M^2B$

(C) $X^T=(M^2B)^T=B^T(M^2)^T=B^TM^2 (as B^T=-B)$

$X^TMB=B^TM^3B=-k(B^TMB)$

Since $M^3=-kM$

Since M is skew-symmetric, so $B^TMB=0$ for any B. (multiply)

Q.63 Let $x = \tan\theta, x > 0, \theta \in (0, \pi/2)$.

$$\frac{1-x^2}{1+x^2} = \cos 2\theta = \sin\left(\frac{\pi}{2} - 2\theta\right)$$

$$\frac{d}{dx}\left(\frac{\pi}{2} - 2\tan^{-1}x\right) = \frac{-2}{1+x^2}$$

$$f'(x) = 2\sin(\pi x) - \frac{2}{1+x^2}$$

(A) $f'(2) = -0.4 < 0$

$$f'(2.5) = 2 - \frac{2}{1+6.25} > 0$$

$f'(x)$ changes in $(2, 2.5) \subset (2, 3)$.

(B) $f(1) = 0$

$$f(2) = -\sin^{-1}\left(\frac{3}{5}\right) - \frac{4}{\pi}$$

(C) $f''(x) = 2\pi\cos\pi x + \dfrac{4x}{(1+x^2)^2}$,

$x \in (0, 1/2) \Rightarrow \cos\pi x > 0$

Since $f''(x) > 0 \Rightarrow f'(x)$ is st. increasing

(D) as $x \to \infty$,

$$\text{term } 1 \to \frac{-\pi}{2}$$

$$\text{term } 2 = \frac{-2}{\pi}(1 + \cos(\pi x))$$

$\uparrow$ oscillates

Q64. $S_n = S_{n-1} + S_{n-2}$ (Fibonacci!!)

$S_k^2 = S_k(S_{k+1} - S_{k-1}) = S_k S_{k+1} - S_k S_{k-1}$

$\sum_{k=1}^{n} S_k^2 = S_n S_{n+1}$ as $S_0 = 0$ (Telescopic)

$$\sum_{k=1}^{10} \binom{10}{k} 2^k \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} \right)$$

$$= \frac{1}{\alpha - \beta} \left[\sum_{k=1}^{10} \binom{10}{k} (2\alpha)^k - \sum_{k=1}^{10} \binom{10}{k} (2\beta)^k \right]$$

$$= \frac{((1+2\alpha)^{10} - 1) - ((1+2\beta)^{10} - 1)}{\alpha - \beta}$$

$$= \frac{(1+2\alpha)^{10} - (1+2\beta)^{10}}{\alpha - \beta}$$

$$\boxed{1 + 2\alpha = \underbrace{1 + \alpha}_{\alpha^2} + \alpha = \alpha + \alpha^2 = \alpha(\alpha+1) = \alpha^3}$$

$$= \frac{(\alpha^3)^{10} - (\beta^3)^{10}}{\alpha - \beta} = \frac{\alpha^{30} - \beta^{30}}{\alpha - \beta} = S_{30}$$

(C) $X = \sum_{n=1}^{\infty} S_n x^n = \frac{x}{1 - x - x^2}$ (generating function)

Differentiate and Put $x = \frac{1}{10}$

$$\boxed{\sum_{n=1}^{\infty} \left(\frac{S_n}{10^n} \right) = \frac{10}{89}}$$

(D) $\frac{S_{n+1}}{S_n} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} = \frac{\alpha - \beta(\beta/\alpha)^n}{1 - (\beta/\alpha)^n}$

$|\beta/\alpha| < 1 \Rightarrow \lim_{n\to\infty} (\beta/\alpha)^n = 0$

$$\lim_{n\to\infty} \frac{S_{n+1}}{S_n} = \alpha = \frac{\sqrt{5}+1}{2}$$

BONUS:

$f(x) = S_0 + S_1 x + S_2 x^2 + S_3 x^3 + \ldots$

$x f(x) = S_0 x + S_1 x^2 + \ldots$

$x^2 f(x) = S_0 x^2 + S_1 x^3 + \ldots$

$f(x)(1 - x - x^2) = S_0 + (S_1 - S_0)x + (S_2 - S_1 - S_0)x^2 + (S_3 - S_2 - S_1)x^3 + \ldots$

$$\Rightarrow f(x) = \frac{x}{1 - x - x^2}$$

As $S_0 = 0, S_1 = 1$ and $S_n - S_{n-1} - S_{n-2} = 0 \; \forall \; n \geq 2$
(That was not required though as use G.P/A.G.P!)

BONUS:

$$\sum_{k=0}^{n} \binom{n}{k} S_k = S_{2n}$$

$$\sum_{k=0}^{n} k \binom{n}{k} = nS_{2n-1}$$

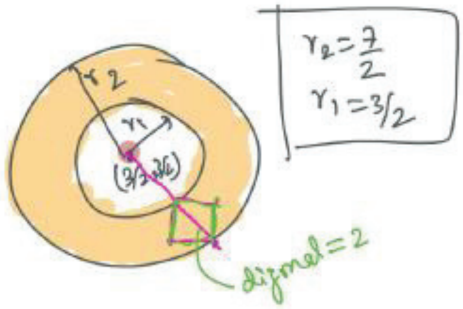
$$\sum_{k=0}^{n} \binom{n}{k} S_k^2 = \frac{S_{2n} S_{2n+1}}{2}$$

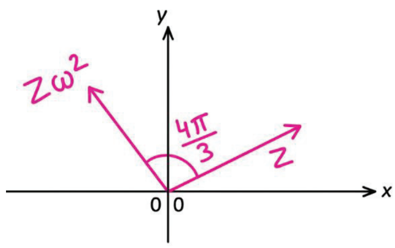
$$\sum_{k=0}^{n} \binom{n}{k}^2 S_k = \frac{1}{\alpha - \beta} \left[\sum_{k=0}^{n} \binom{n}{k}^2 \alpha^k - \sum_{k=0}^{n} \binom{n}{k}^2 \beta^k \right]$$

Is coefficient of x^n in

$$\frac{1}{\alpha - \beta} \left[(1 + \alpha x)^n (x+1)^n - (1 + \beta x)^n (1 + x)^n \right]$$

Q.65





$z = re^{i\theta}$

$$\text{Re}(\omega^2 z) = 0 \Rightarrow \theta + \frac{4\pi}{3} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\boxed{\theta = \frac{11\pi}{6} \text{ or} - \frac{5\pi}{6}}$$

$$\text{Since} |z| = \frac{3}{\sqrt{2}} \Rightarrow \boxed{z = \frac{3}{\sqrt{2}} e^{i\pi/6}}$$

Q.66

$$f(x)+e^{-x}f(e^{-x})=x+e^{-2x}$$

$$\int_0^a f(x)dx+\int_0^a e^{-x}f(e^{-x})dx=\int_0^a (x+e^{-2x})dx$$

where $ae^a=1$.

$$\int_0^a f(x)dx+\int_a^1 f(x)dx=\int_0^a (x+e^{-2x})dx$$

$$\Rightarrow \boxed{\int_0^1 f(x)dx=\left(\frac{a^2}{2}-\frac{a^2}{2}\right)-(0-1/2)=1/2}$$

(B) $f'(x)+e^{-x}(-1)f(e^{-x})+e^{-x}f'(e^{-x})(-e^{-x})=1$

at x = a we have

$\boxed{f'(a)=1}$ as $\boxed{ae^a=1}$

$$g'(a)=\frac{1}{f'(g(a))}=\frac{1}{f'(a)}=1$$

Since $f(a)+e^{-a}f(e^{-a})=a+e^{-2a}$ (using $e^{-a}=a$)

$\Rightarrow f(a)(1+a)=a(1+a) \Rightarrow \boxed{f(a)=a}$

$\boxed{f''(a)\left(\frac{1+a^3}{a}\right)=0} \Rightarrow f''(a)=0$

Q.67

$$y^{-1/2}\frac{dy}{dx}-2y^{1/2}e^{x}=2e^{x}$$

$$v=y^{1/2}\rightarrow\frac{dv}{dx}=\frac{1}{2}y^{-1/2}\frac{dy}{dx}$$

$$\Rightarrow\frac{dv}{dx}-e^{x}v=e^{x}$$

$$\text{I.F.}=e^{\int -e^{x}dx}=e^{-e^{x}}$$

$$\Rightarrow \boxed{\sqrt{y}=ce^{e^{x}}-1}\ \text{Put } \phi(0)=1$$

$$\phi(x)=(2e^{e^{x}-1}-1)^{2}$$

(A) $x\rightarrow\infty, e^{e^{x}-1}\rightarrow\infty$

(D) $f(x)=2e^{e^{x}-1}-1 \quad \forall\, x\in(0,\infty)$

$e^{x}>1\Rightarrow e^{e^{x}-1}>1$ So, $f\left(x\right)>0$

Let $\phi(n)=(f(x))^{2}$

$$\phi'(n)=2ff'=2f(2e^{e^{x}-1}\cdot e^{x})$$

for $x\in(0,\infty)>0$

Q.68

$$\lim_{x\to0}\frac{a\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots\right)+b\left(x-\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right)+c\left(x+\frac{x^3}{6}+\frac{9x^5}{5!}+\ldots\right)}{x^5}=3$$

$$=\lim_{x\to0}\frac{(a+b+c)x+\left(\frac{a}{3}-\frac{b}{3}+\frac{c}{6}\right)x^3+\left(\frac{2a}{15}+\frac{b}{5}+\frac{9c}{5!}\right)x^5+\ldots}{x^5}$$

$$a+b+c=0$$

$$2a-2b+c=0$$

$$\frac{2a}{15}+\frac{b}{5}+\frac{3c}{40}=3$$

$$a=30,\quad b=10,\quad c=-40$$

Logic:

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\ldots\quad(|x|<1)$$

$$\int_0^x\frac{1}{1+t^2}dt=\int_0^x(1-t^2+t^4-t^6+\ldots)dt$$

$$\Rightarrow\ \boxed{\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots}$$

$$\boxed{\int_0^x(1-t^2)^{-1/2}dx=\sin^{-1}x}\ \downarrow$$

$$(1-x^2)^{-1/2}=1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\ldots$$

$$1+nx+\frac{n(n-1)}{2!}x^2+\ldots=(1+x)^n$$

$$\Rightarrow\ \boxed{\sin^{-1}x=x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^7}{7}+\ldots}$$

$$\frac{d}{dx}(\tan x)=\sec^2x=\frac{1}{\cos^2x}$$

$$\cos x\approx1-\frac{x^2}{2!}+\frac{x^4}{4!}$$

$$\cos^2x\approx(1-\frac{x^2}{2}+\frac{x^4}{24})^2\approx1-x^2+\frac{x^4}{3}$$

$$\sec^2x\approx\frac{1}{1-(x^2-\frac{x^4}{3})}\approx1+(x^2-\frac{x^4}{3})+\ldots\approx1+x^2+\frac{2}{3}x^4$$

$$\int_0^x\sec^2t\,dt\approx\int_0^x(1+t^2+\frac{2}{3}t^4+\ldots)dt$$

$$\boxed{\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots}$$

$\downarrow$

$$\frac{\sin x}{\cos x} = \sin x(\cos x)^{-1}$$

$$\approx (x - \frac{x^3}{6} + \frac{x^5}{120})(1 - \frac{x^2}{2} + \frac{x^4}{24})^{-1}$$

$$\approx (x - \frac{x^3}{6} + \frac{x^5}{120})(1 + \frac{x^2}{2} + \frac{5}{24}x^4)$$

$$(1-u)^{-1} = 1 + u + u^2 + \ldots$$

$$u = \frac{x^2}{2} - \frac{x^4}{24} \Rightarrow u^2 \approx \frac{x^4}{4}$$

$$\approx x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

Q.69

$$x^2 = 4y$$

$$x^2 + (y - y_n)^2 = r_n^2$$

D=0, gives

$$y_n = \frac{r_n^2}{4} + 1$$

$y_{n+1} - y_n = r_{n+1} + r_n$ (tangent condition)

$$\Rightarrow r_{n+1} - r_n = 4$$

$r_0 = \frac{r_0^2}{4} + 1$ as $y_0 = r_0 \Rightarrow r_0 = y_0 = 2$

$r_n = 2 + 4n \rightarrow$ forms **A.P.** (Arithmetic Progression)

(A) $\frac{r_n^2 - r_{n-1}^2}{y_n - y_{n-1}} = r_n - r_{n-1} = \mathbf{4}$

(B) $y_n - y_{n-1} = r_n + r_{n-1} = 4n + 2 + 4n - 2 = \mathbf{8n}$

(C) $\sum_{k=1}^{10} \frac{y_k - y_{k-1}}{k} = \sum_{k=1}^{10} 8 = 80$

(D) $|r_{25} - r_{10}| = |(25-10) \times 4| = 60$

Q.70

Let $u = \dfrac{x}{x+1} \Rightarrow x = \dfrac{u}{1-u}$

$dx = \dfrac{du}{(1-u)^2}$

$$S(m,n) = \int_0^{1/2} \frac{\left(\dfrac{u}{1-u}\right)^m \left(\dfrac{1-2u}{1-u}\right)^n}{(1-u)^{m+n+2}} \cdot \frac{du}{(1-u)^2}$$

$$\boxed{S(m,n) = \int_0^{\frac{1}{2}} u^m (1-2u)^n \, du}$$

Put t=2u

$$S(m,n) = \frac{1}{2^{m+1}} \int_0^1 t^m (1-t)^n \, dt$$

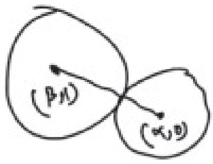
$$\Rightarrow S(m,n) = \frac{I(m,n)}{2^{m+1}}$$

$$S(2,2) = \frac{I(2,2)}{8} = \frac{\int_0^1 x^2 (1-x)^2 \, dx}{8} = \mathbf{\frac{1}{240}}$$

BONUS:

$$\left| \frac{\int_0^1 x^{7/2} (1-x)^{9/2} \, dx}{\int_0^1 \dfrac{x^{7/2} (1-x)^{9/2}}{(x+5)^{10}} dx} \right| = 5^{\frac{11}{2}} 6^{\frac{9}{2}}$$

Q.71



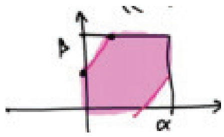
$|\alpha-\beta|<\sqrt{3}$ (intersects)

$|\alpha-\beta|=\sqrt{3}$ (touch)

$|\alpha-\beta|>\sqrt{3}$ (disjoint)

$S=\{(\alpha,\beta):|\alpha-\beta|\le\sqrt{3},\alpha,\beta\in[0,2]\}$

$T=\{(\alpha,\beta):\alpha,\beta\in[0,2]\}$



$\text{Probability}=\dfrac{\text{area}(S)}{\text{area}(T)}$

$=\boxed{\dfrac{4\sqrt{3}-3}{4}}$

Q.72

$L_t: y=1-\dfrac{x}{t}, y=x^2$ gives

$x=\dfrac{-1+\sqrt{1+4t^2}}{2t}=\dfrac{2t}{1+\sqrt{4t^2+1}}$

$\underset{\substack{\downarrow\\ \text{evenfn}}}{f(t)}=\dfrac{4t^2}{4t^2+2+2\sqrt{4t^2+1}}, t\ge 0$

(A) $f(t)\in[0,1)$

(B) f is continuous

(C) $\lim\limits_{t\to\infty} f(t)=1$

(D) $f'(0)=\lim\limits_{t\to 0}\dfrac{f(t)}{t}=0$

Q.73

M-1:

A_n: Good numbers (Divisible by 3)

B_n: Bad numbers (Not divisible by 3)

$A_{n+1} = 2A_n + B_n$ $\quad$ $B_n + A_n = 4^n$

{3, a} unique way

Hence $\boxed{A_{n+1} - A_n = 4^n} \Rightarrow \boxed{\mathbf{A_n = \frac{4^n + 2}{3}}}$

M-2

$f(x) = (x^2 + x^3 + x^7 + x^9)^n$

$\frac{f(1) + f(\omega) + f(\omega^2)}{3} = \frac{4^n + 2}{3}$ (as sum is divisible by 3)

(Here ω is cube root of unity: $\omega^3 = 1, \omega \neq 1$)

Q.74

Let $z = e^{i\theta}, \theta \in [0, 2\pi)$

$|1 + e^{i\theta}| = \left|e^{i\theta/2}(e^{-i\theta/2} + e^{i\theta/2})\right| = |2\cos\theta/2|$

$|1 - z + z^2| = |z(\bar{z} - 1 + z)| = |2\cos\theta - 1|$

Let $t = |\cos\theta/2|, t \in [0,1]$

$E(t) = 2t + |4t^2 - 3|$

$\max(s) = \frac{13}{4}$ (

Q.75

$|a + b\omega + c\omega^2| = |(a-c) + (b-c)\omega| = |p + q\omega|$

where p and q are integers (infinite a, b, c)

$|a\omega + b| = 1 \Rightarrow (a-b)^2 + ab = 1$

$(0,\ 1),\ (1,\ 0),\ (1,\ 1),\ (-1,\ -1),\ (0,\ -1),\ (-1,\ 0).$

Q.76

$$\int_0^1\left(e^{\frac{ax}{2}}f(x)-e^{-\frac{ax}{2}}\right)^2dx=-\frac{(a-2)^2}{4a^2}\le 0$$

$\Rightarrow f(x)=e^{-ax}$ with $a=2$

This method of boundary case is very popular! Equality case in A.M.-G.M. etc.

Q.77

[A] $h(t)=|\sin(5\pi t)-\sin(3\pi t)|$ is a modulus of continuous fn hence continuous and integral of continuous fn is differentiable.

So f(x) is differentiable for all $x\in(0,1)$.

[B] $f'(x)=|\sin(5\pi x)-\sin(3\pi x)|$

|u| is not differentiable at u=0 and $u'\ne 0$

$\Rightarrow \sin(5\pi x)-\sin(3\pi x)=0$

$\Rightarrow 2\cos(4\pi x)\sin(\pi x)=0$

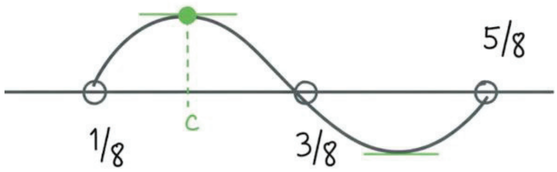
$$\underbrace{\sin(\pi x)}_{\text{No soln}}$$

$x\in\left\{\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}\right\}$ (Sharp corners as $u'\ne 0$)

[C] f' is continuous in $[\frac{1}{8},\frac{3}{8}]$ and differentiable in $(\frac{1}{8},\frac{3}{8})$.

$f'(\frac{1}{8})=|\sin(\frac{5\pi}{8})-\sin(\frac{3\pi}{8})|=0$

$f'(\frac{3}{8})=0$



(D) Since |sin A -sin B| ≤ |A - B| (Geometrically chord!)

$$|\sin(5\pi t)-\sin(3\pi t)|\leq|5\pi t-3\pi t|\leq 2\pi t$$

(Strict inequality as t > 0 because sin x < x for x > 0)

$$f(x)=\int_0^x |\text{�}|\,dt<\int_0^x 2\pi t\,dt=\pi x^2$$ ☺

So, $\boxed{f(x)<\pi x^2 \text{ for } x\in(0,1/8)}$

$$\int_0^x f(t)dt<\int_0^x \pi t^2 dt=\frac{\pi x^3}{3}$$

Q.78

$$S_A = 5x + 7x^2 + 9x^3 + \ldots$$

$$xS_A = 5x^2 + 7x^3 + \ldots$$

$$(1-x)S_A = 5x + \underbrace{(2x^2 + 2x^3 + \ldots)}_{\text{19 terms}}$$

$$\boxed{\bar{A} = \frac{3}{20} - \left(\frac{23}{20}\right)3^{-20}}$$

$$c_n = 3 - \frac{2}{n^2+2}$$

$$d_n = 4 - \frac{3}{n^2+2}$$

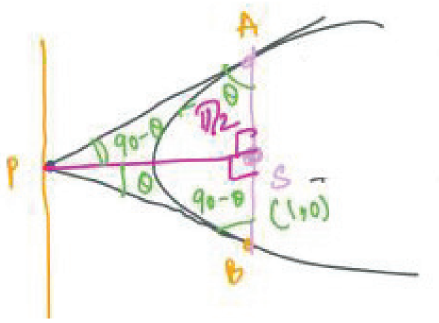
Let $X = \dfrac{1}{n^2+2} \Rightarrow \begin{matrix} \text{set } C \to 3 - 2X \\ \text{set } D \to 4 - 3X \end{matrix}$

$$\sigma_C = |-2|\,\sigma_X = 2\sigma_X$$

$$\sigma_D = |-3|\,\sigma_X = 3\sigma_X$$

$\boxed{3\sigma_C = 2\sigma_D}$ (No need to calculate)

Q.79 SA.SB=SP^2(From similarity)



playlist (pure geometry/conic sections)

$$\sin\theta = \frac{SP}{PA}$$

$$\Rightarrow \frac{1}{PA^2} + \frac{1}{PB^2}$$

$$\cos\theta = \frac{SP}{PB} = \frac{1}{SP^2}$$

Area of ΔPAB

$$\frac{1}{2}PA \cdot PB = \frac{1}{2} \times AB \times SP$$

$$\boxed{\Rightarrow \mathbf{PA \cdot PB = (AB)SP}}$$

$$\text{Slopeof } (AB) = 1,\ AB = 4a\csc^2\theta$$

$$= \frac{4}{\sin^2\theta} = \frac{4}{\sin^2\frac{\pi}{4}} = 8$$

$$PA \cdot PB = 8SP$$

Important: If Point is on Directrix

$$\frac{1}{SA} + \frac{1}{SB} = \frac{1}{a}$$

$$SP^2 = SA \cdot SB$$

$$PA \cdot PB = AB \cdot SP = \frac{(SP)^3}{a}$$

$$\frac{1}{PA^2} + \frac{1}{PB^2} = \frac{1}{SP^2}$$

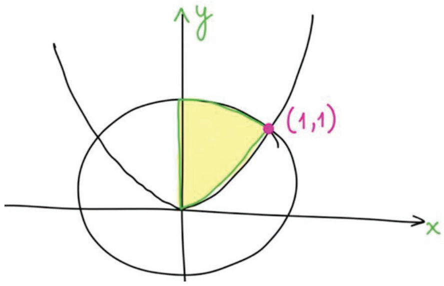
$$AB = \frac{SP^2}{a}$$

$$\Delta(PAB) = \frac{(SP)^3}{2a}$$

$$\frac{r(A)}{r(B)} = 5\sqrt{2} - 7 \text{ or } 5\sqrt{2} + 7$$

as per labelling of A or B

Q.80



$$A(\Omega) = \int_0^1 (\sqrt{2-x^2} - x^2)dx = \frac{3\pi + 2}{12}$$

(A) and (B) is obvious from continuity and I.V.T.

$$A(Y_m) = \frac{3\pi + 2}{24}$$

$$A(Y_m) = \frac{1}{2}r^2\theta = \frac{1}{2}(\sqrt{2})^2(\pi/2 - \tan^{-1} m) = \cot^{-1}(m)$$

$$\boxed{m = \cot\left(\frac{\pi}{8} + \frac{1}{12}\right)}$$

For m > 1 the line lies above the parabola inside Ω.

The green region becomes circular arc of radius $\sqrt{2}$.

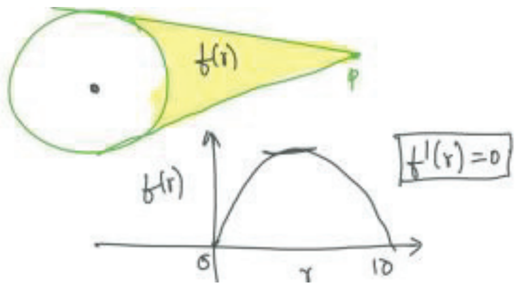
Q.81

$$\boxed{\begin{aligned} r_1 &= \frac{5}{3} \\ r_2 &= \frac{5}{9} \\ &\vdots \\ r_n &= \frac{5}{3^n} \end{aligned}} \Rightarrow S = \pi(\frac{5}{3})^2 \times \frac{1}{1-1/9} = \frac{25\pi}{8}$$

$$\boxed{\frac{\text{Sum of area}}{\text{Total area}} \cong 57.3\%}$$

(A)

$$\boxed{f'(r) = 0}$$



(A)

(B) $\Delta = \dfrac{r(L^2 - r^2)^{3/2}}{L^2} = 100\sin\theta\cos^3\theta$

$\dfrac{d\Delta}{d\theta} = 0 \Rightarrow \boxed{\theta = \dfrac{\pi}{6}}$

$\Rightarrow r = 10\sin\dfrac{\pi}{6} = 5$

(C) $r_1 = r\left(\dfrac{1-\sin\theta}{1+\sin\theta}\right) = 10\sin\theta\left(\dfrac{1-\sin\theta}{1+\sin\theta}\right)$

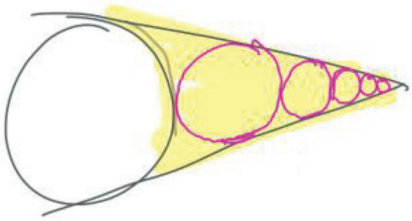
$g(s) = \dfrac{10(s-s^2)}{1+s}, \quad s = \sin\theta$

$g'(s) = s^2 + 2s - 1 = 0$

$\boxed{\sin\theta = \sqrt{2} - 1}$

$\boxed{r_{max} = 10(3 - 2\sqrt{2})}$

(Think of $r = \dfrac{A}{s}$ for smart way!)



(D)

$$\boxed{A_{total} = 25\sqrt{3} - \frac{25\pi}{3}}$$

Q.82

$$u = xt \Rightarrow dt = \frac{du}{x}$$

$$H(x) = \int_{x^2}^{x^3} f(u)\frac{du}{x} = \frac{1}{x}\int_{x^2}^{x^3} (2u - \sin u)du$$

$$= \frac{x^6 + \cos x^3 - (x^4 + \cos x^2)}{x}$$

$$\lim_{x\to 0^+} \frac{H(x)}{x^3} = \lim_{x\to 0^+} \frac{x^6 - x^4 + \cos x^3 - \cos x^2}{x^4}$$

$$= \lim_{x\to 0^+} \frac{-2\sin\left(\frac{x^3 + x^2}{2}\right)\sin\left(\frac{x^3 - x^2}{2}\right)}{x^4}$$

$$= -\frac{1}{2}$$

$$(g^{-1})'(0) = \frac{1}{g'(g^{-1}(0))} = \frac{1}{g'(0)} = (f'(0))^3 = 1$$

Generalized Leibniz

$$\boxed{\begin{array}{l} H(x) = \int_{f(x)}^{g(x)} h(x,t)dt \\ H'(x) = h(g(x),x)g' - h(x,f(x))f' + \int_{f(x)}^{g(x)} \frac{\partial}{\partial x}h(x,t)dt \end{array}}$$

$$H(x) = \int_{x}^{x^2} f(xt)dt$$

$$H'(x) = f(x^3) \times 2x - f(x^2) + \int_{x}^{x^2} tf'(xt)dt$$

$$H'(x) = 2f(1) - f(1) + \int_{1}^{1} tf'(xt)dt$$

$$= f(1) = 2 - \sin 1$$

Q.83

$\boxed{a^2+b^2=13}$ as $\boxed{c^2=a^2m^2+b^2}$

(A) $(2,3)$ and $(3,2)\in\mathbb{N}\times\mathbb{N}$

(B) $x^2+y^2=13$ (Director circle)

(C) $ab\le\dfrac{a^2+b^2}{2}\Rightarrow A_{\max}=\dfrac{13\pi}{2}$

(D) $P(x_0,y_0)=(\dfrac{a}{e},0)$

$$x_0=\frac{a}{\sqrt{1-\frac{b^2}{a^2}}}=\frac{a^2}{\sqrt{a^2-b^2}}$$

Since $\boxed{a^2+b^2=13}\Rightarrow\boxed{\lim\limits_{b\to 0^+}(x_0)=\sqrt{13}}$

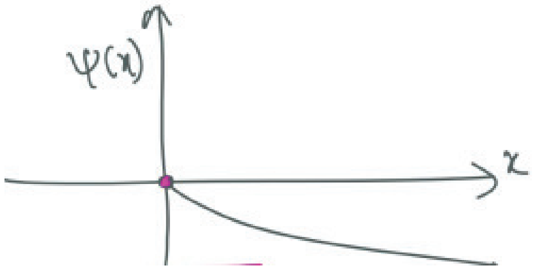
Q.84

$f'(x)-cf(x)\le 0$

$\Rightarrow dxd(f(x)e-cx)\le 0$

$\Rightarrow fe^{-cx}$ is $\downarrow$

Let $\psi(x)=f(x)e^{-cx}\le 0\quad\forall x\in\mathbb{R}$



But $\boxed{f(x)\ge 0}\quad\forall x\in\mathbb{R}.$

Hence, No such c exists.

Q.85

$$\| s \| = \int_0^1 \sin(\pi x)dx = \frac{2}{\pi} = \lim_{n\to\infty} \| R_n \|$$

$(x+y)^4 = x^2 \Rightarrow x+y = \sqrt{x} \Rightarrow y = \sqrt{x} - x \rightarrow$ PARABOLA

$$\| g \| = \int_0^1 (\sqrt{x} - x)dx = \frac{1}{6}$$

BONUS: If G_n is geometric mean of $\binom{n}{k}, k \in \{0,1,\dots,n\}$ Then $\lim_{n\to\infty}(G_n)^{1/n}$ is:

WORKING:

$$\binom{n}{0}\binom{n}{1}\cdots\binom{n}{n} = \prod_{k=0}^{n} \frac{n!}{k!(n-k)!}$$

$$= \frac{(n!)^{n+1}}{(1!2!\cdots n!)^2}$$

$$= \prod_{k=1}^{n} (n+1-k)^{n+1-2k}$$

$$= \prod_{k=1}^{n} \left(\frac{n+1-k}{n+1}\right)^{n+1-2k}$$

Since

$$\sum_{k=1}^{n}(n+1-2k) = 0, \sum_{k=1}^{n}(n+1)^{n+1-2k} = 1,$$

$$G_n = \prod_{k=1}^{n}\left(1 - \frac{k}{n+1}\right)^{1-\frac{2k}{n+1}}$$

$$\Rightarrow \frac{1}{n}\ln G_n = \frac{1}{n}\sum_{k=1}^{n}\left(1 - \frac{2k}{n+1}\right)\ln\left(1 - \frac{k}{n+1}\right)$$

$$\lim_{n\to\infty} \frac{\ln G_n}{n} = \int_0^1 (1-2x)\ln(1-x)dx$$

$$= 2\int_0^1 (1-x)\ln(1-x)dx - \int_0^1 \ln(1-x)dx$$

$$= \frac{1}{2}$$

$\boxed{\lim_{n\to\infty}(G_n)^{\frac{1}{n}} = \sqrt{e}}$ (Practice Riemann Sum for PYPS!)

Q.86

$$P(x=0)=\prod_i(1-p_i)=1-\sum_i p_i+\sum_{i\neq j}p_ip_j-p_1p_2p_3=\frac{2}{5}$$

$$P(x=1)=\sum_{\{i,j,k\}=\{1,2,3\}}p_i(1-p_j)(1-p_k)$$

$$=\sum_i p_i-2\sum_{i\neq j}p_ip_j+3p_1p_2p_3=\frac{13}{30}$$

$$P(x=2)=\sum_{\{i,j,k\}=\{1,2,3\}}p_ip_j(1-p_k)$$

$$=\sum_{i\neq j}p_ip_j-3p_1p_2p_3=\frac{3}{20}$$

$$P(x=3)=p_1p_2p_3=\frac{1}{60}$$

This is Linear system in unknown, $\sum_i p_i, \sum_{i\neq j}p_ip_j$ and $p_1p_2p_3$ with Soln

$$\sum_i p_i=\frac{47}{60},\quad \sum_{i\neq j}p_ip_j=\frac{1}{5},\quad p_1p_2p_3=\frac{1}{60}$$

It follows p_1,p_2,p_3 are roots of

$$\boxed{x^3-\frac{47}{60}x^2+\frac{1}{5}x-\frac{1}{60}=0}$$

p_1,p_2,p_3 is $\frac{1}{3},\frac{1}{4}$ and $\frac{1}{5}$ in some order

M-2

Use scheme of poisson!

$$(p_1x+1-p_1)(p_2x+1-p_2)(p_3x+1-p_3)$$

$$=\frac{2}{5}+\frac{13}{30}x+\frac{3}{20}x^2+\frac{1}{60}x^3$$

$$1-\frac{1}{p_i}, i=1,2,3$$

$x^3+9x^2+26x+24=0$ and $p_1p_2p_3=\frac{1}{60}$

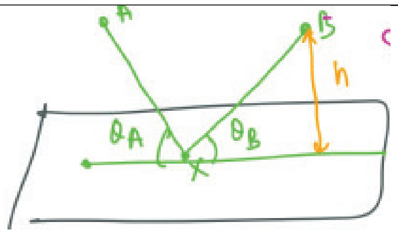
The roots -2, -3, -4 gives $\{\frac{1}{3},\frac{1}{4},\frac{1}{5}\}$ as p_i.

Atleast two pass: $P(X\geq 2)=\frac{3}{20}+\frac{1}{60}=\frac{1}{6}$

$p_1\neq p_2\neq p_3$ (Not equally likely)

$$\sum xp(x) = 0\cdot\frac{24}{60} + 1\cdot\frac{26}{60} + 2\cdot\frac{9}{60} + 3\cdot\frac{1}{60} = \frac{47}{60}$$

Q.87



$$\sin\theta = \frac{h}{\text{hypotenuse}}$$

$$h_A = \frac{1}{\sqrt{3}} \text{ and } h_B = \frac{|r-1|}{\sqrt{3}}$$

$$\theta_A = \theta_B \Rightarrow \frac{h_A}{AX} = \frac{h_B}{BX} \Rightarrow \boxed{\frac{AX}{BX} = \frac{h_A}{h_B}}$$

$$\boxed{\frac{h_A}{h_B} = \frac{1}{|r-1|} = \lambda}$$

HINT If $\lambda = 1$, locus AX = BX is a perpendicular bisector of plane segment.

The bisector plane of A and B is $y = \frac{r}{2}$ since they differ only in y-coordinate.

Since |r - 1| = 0 is true for $\boxed{r = 0, 2}$

So, straight line.

If $\lambda \neq 1$ locus AX = λ BX is apollonian sphere.

$$(x-1)^2 + y^2 + (z-1)^2 = k((x-1)^2 + (y-r)^2 + (z-1)^2)$$

where $k = \lambda^2 = \frac{1}{(r-1)^2}$.

Rearranging:

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = R_s^2$$

$$C_s : (1, \frac{-rk}{1-k}, 1)$$

$$R_s : \frac{|r|\sqrt{k}}{|1-k|}$$

The locus is intersection of sphere with plane. Let d be distance of Cs from x + y - z - 1 = 0

for $r = 1 + \sqrt{3}$, $\boxed{k = \frac{1}{3}}$

$d = \frac{\sqrt{3}+1}{2}$ and $R_s = \frac{3+\sqrt{3}}{2}$

$\boxed{d < R_s} \rightarrow$ circle

Similarly for r = 4, (Sphere) $\cap$ (Plane) = circle

$$\boxed{\begin{aligned} \text{Sphere} \cap \text{Plane} &= \ (\text{point}) \\ &= \bigcirc\ (\text{circle}) \\ &= \phi\ (\text{empty set}) \end{aligned}}$$

$\rightarrow$ for which this happens! (Do not be surprised!)

Q.88

$\boxed{S=2\sqrt{2}} \rightarrow$ ellipse direct def.

$$\text{Area}(A_1)=\frac{1}{2}\times 2(y_0-y_2)=y_0-y_2$$

$$r_1=\frac{\Delta_1}{s}=\frac{y_0-y_2}{2\sqrt{2}}$$

$$r_2=\frac{y_0-y_1}{2\sqrt{2}}$$

$$\Rightarrow \boxed{r_1-r_2=\frac{y_1-y_2}{2\sqrt{2}}}$$

Using the property chords passes through $(\pm 1, 0)$.

$$y_1=\frac{-y_0}{3+2x_0},\ y_2=\frac{-y_0}{3-2x_0} \Rightarrow f(x_0)=\frac{4x_0\sqrt{1-\frac{x_0^2}{2}}}{9-4x_0^2}.$$

$$y_1-y_2=\frac{4x_0y_0}{9-4x_0^2}$$

let $x_0^2=t$ So,

$$G(t)=\frac{8(2t-t^2)}{(9-4t)^2}$$

$$\frac{dg}{dt}=0 \Rightarrow t=\frac{9}{5} \Rightarrow y_1-y_2=\frac{2\sqrt{2}}{3}$$

$$\boxed{\max(r_1-r_2)=\frac{1}{3}}$$

Q.89

$$A(b)=\int_0^1 |x^2-b|\,dx$$

$$=\int_0^{\sqrt{b}}(b-x^2)\,dx+\int_{\sqrt{b}}^1(x^2-b)\,dx$$

$$=\frac{4}{3}b^{3/2}-b+\frac{1}{3}$$

$$A'(b)=0 \Rightarrow b_0=\frac{1}{4}$$

$$A\left(\frac{1}{4}\right)=\frac{1}{4}$$

Q.90

$$\boxed{\begin{array}{l} S_{n+1} = S_n + C_n \\ C_{n+1} = C_n - 2S_n \end{array}}$$

$$S_n, C_n \in \mathbb{Z}$$

Find characteristics equation

$$x = \sqrt{3}e^{i\theta}$$

$$\Rightarrow x = 1 + i\sqrt{2}$$

$$x^2 - 2x + 3 = 0$$

$$a_{n+2} - 2a_{n+1} + 3a_n = 0 \quad ((\sqrt{3}e^{i\theta})^n \text{ for this})$$

$$S_n = \mathrm{Im}\left(\frac{(1+i\sqrt{2})^n}{\sqrt{2}}\right)$$

$$\sum_{n=0}^{\infty}\frac{S_n}{3^n} = \mathrm{Im}\left(\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\left(\frac{1+i\sqrt{2}}{3}\right)^n\right)$$

$$= \mathrm{Im}\left(\frac{3(2+i\sqrt{2})}{6\sqrt{2}}\right) = \frac{1}{2}$$

OR

$$z = \sqrt{3}(\cos\theta + i\sin\theta)$$

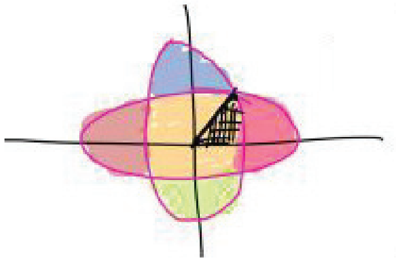
$$z^n = (\sqrt{3})^n(\cos(n\theta) + i\sin(n\theta)) = c_n + i\sqrt{2}s_n$$

$$\sum_{n=0}^{\infty}\frac{z^n}{3^n} = \sum_{n=0}^{\infty}\left(\frac{1+i\sqrt{2}}{3}\right)^n = 1 + \frac{\sqrt{2}}{2}i$$

$$\text{Real part } \sum\frac{c_n}{3^n} = 1$$

$$\sqrt{2}\sum\frac{s_n}{3^n} = \frac{\sqrt{2}}{2} \Rightarrow \boxed{\sum\frac{s_n}{3^n} = 1/2}$$

Q.91



$A(E_1) = 3A$

$A(R_0) = \frac{\pi ab}{3}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in polar form :

$r^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}$

$\text{Area}(R_0) = 8\int_0^{\pi/4} \frac{1}{2} r^2 d\theta$

$= 4a^2b^2 \int_0^{\pi/4} \frac{\sec^2\theta d\theta}{b^2 + a^2\tan^2\theta}$

$= 4a^2b^2 \int_0^1 \frac{dt}{b^2 + a^2t^2}$

$= 4ab\tan^{-1}\left(\frac{a}{b}\right)$

Since, $A(R_0) = \frac{1}{3}\pi ab$

$\Rightarrow 4ab\tan^{-1}(b/a) = \frac{\pi ab}{3}$

$\Rightarrow \frac{a}{b} = 2 + \sqrt{3}$

Q.92 Let P = (h, k). The combined equation of tangents from (h, k) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)-\left(\frac{xh}{a^2}+\frac{yk}{b^2}-1\right)^2=0$$

Any conic that can be drawn through the points of intersection of these tangents with the coordinate axes is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)-\left(\frac{xh}{a^2}+\frac{yk}{b^2}-1\right)^2+\lambda xy=0$$

or

$$x^2\left(\frac{k^2}{a^2b^2}-\frac{1}{a^2}\right)+y^2\left(\frac{h^2}{a^2b^2}-\frac{1}{b^2}\right)+xy\left(\lambda-\frac{2hk}{a^2b^2}\right)+\frac{2xh}{a^2}+\frac{2yk}{b^2}-\frac{h^2}{a^2}-\frac{k^2}{b^2}=0$$

Locus $x^2-y^2=a^2-b^2$

$a\neq b\rightarrow$ Hyperbola

$a=b\rightarrow$ p.o.lines.

NUMERICAL VALUE

Q.93

Total number of permutation matrices = 6.

Type 1: (Identity: I) $\rightarrow$Contribution $: |I|\,tr(I) = 3$

Type 2: Matrices where exactly two rows are swapped. ($A^2 = I$).

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$\Rightarrow |A| = -1(\text{Singleswap}),$

$A^{2026} = (A^2)^{1013} = I$

$\text{Contribution} = (-1)\times 3\times 3 = \mathbf{-9}(3\text{matrices})$

Type 3: Cyclic shift of all three rows; ($A^3 = I$)

$R_1 \rightarrow R_2 \rightarrow R_3 \rightarrow R_1$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$\Rightarrow A^{2026} = A \Rightarrow \text{trace}(A) = 0$

$\Rightarrow \boxed{\text{Contribution} = 0}$

$$\Rightarrow \left|\sum_{A\in M} \psi(A)\right| = |1-6| = 6$$

Q.94

Since $n \geq 1$, we have $x_n \in (0, 1/3]$.

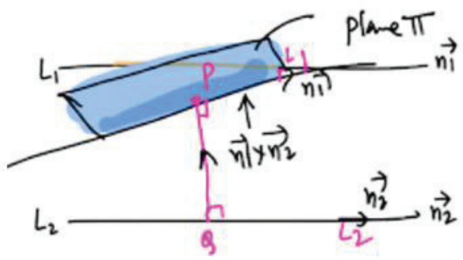
So x_n lies in(0, 1). Let $x = \tan\theta, \theta \in (0, \pi/4)$.

$f(x) = 3\tan^{-1}x \Rightarrow f(x_n) = 3\tan^{-1}x_n$.

So sum is $3\left(\sum_{n=1}^{\infty} \tan^{-1}(n+1) - \tan^{-1}(n)\right) = \frac{3\pi}{4}$

Q.95

$P = (1, -3, 6)\ Q \equiv (7, 3, 4)$



$\overrightarrow{PQ} = (6, 6, -2)$ is normal Equation of plane Π is $3x + 3y - z + 12 = 0$

$d = \frac{12}{\sqrt{19}} \Rightarrow 19d^2 = 144$

Q.96

$$A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$X^T AX = 0$ gives A is skew symmetric. So,

$$A=\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \Rightarrow A^3 = -A(a^2+b^2+c^2) \text{ (Caley-Hamilton)}$$

$\Rightarrow a^2+b^2+c^2=1$

So, $\boxed{a=\pm 1, b=0, c=0}$ and similarly

$|\mathbf{A}+\mathbf{I}|=2$

For all $A \in S$. Hence $\sum_{A\in S}|A+I|=12$

Q.97

$Z = 4e^{i\theta}, \quad \frac{1}{Z}=\frac{1}{4}e^{-i\theta}$

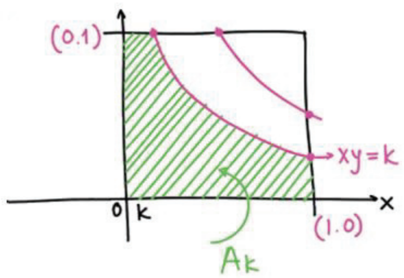
$\text{Re}(\omega)=216\cos\theta-288\sin\theta$

$\max(\text{Re}(\omega))=\sqrt{(216)^2+(-288)^2}$

$=72\sqrt{3^2+4^2}$

$=72\times 5=360$

Q.98



$A(k)=k-k\ln k$

$A=A\left(\frac{1}{2}\right)-A\left(\frac{1}{4}\right)$

=1/4

Q.99

$$ax\sec\theta - by\csc\theta = a^2b^2$$
$$\Rightarrow b^2\cos^2\theta = a^2e^2$$
$$\cos^2\theta = \frac{a^2e^2}{b^2}$$
$$\sin\theta = \frac{1}{\sqrt{2}}$$
$$\Rightarrow \theta = 30^\circ$$
$$\Rightarrow \frac{SM}{S'N} = \frac{SP}{S'P}$$
$$= \frac{e(a/e - x_1)}{e(a/e + x_1)}$$
$$= \frac{a - ex_1}{a + ex_1} = \frac{a - ea\cos\theta}{a + ea\cos\theta} = \frac{1 - e\cos\theta}{1 + e\cos\theta}$$
$$= \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$$
$$= \frac{\sqrt{2}-1}{\sqrt{2}+1} + \frac{\sqrt{2}+1}{\sqrt{2}-1}$$
$$= \frac{(\sqrt{2}-1)^2 + (\sqrt{2}+1)^2}{(\sqrt{2})^2 - 1^2} = \frac{6}{2-1} = 6$$

Q.100

$A^{n+1} = A^n \cdot A$ gives

$\boxed{a_{n+1} = a_n + b_n} \rightarrow a_1 = 1, b_1 = 1$ Fibonacci

$$\Rightarrow L = 1 + \frac{1}{L}$$
$$\Rightarrow L^2 = L + 1 \Rightarrow \left| L^2 - \frac{1}{L} \right| = | L + 1 - L + 1 | = 2$$

Q.101

$$\int_0^1 (f - f^2 - 1/4)dx = 0$$
$$\Rightarrow \int_0^1 (f^2 - 2\cdot\frac{1}{2}\cdot f + (1/2)^2)dx = 0$$

$\Rightarrow \int_0^1 (f - 1/2)^2\,dx = 0 \Rightarrow \boxed{f(x) = 1/2} \rightarrow$continuity

Q.102

$$X = \{(-1,2),(0,-1),(0,1),(0,2),(0,3),(1,2)\}$$
$$P(E) = \frac{|A|}{|S|} = \frac{10}{20} = 1/2$$
$$|A| = N_{total} - N_{collinear} - N_{odd\ area}$$
$$= 20 - 1 - 9 = 10$$

Q.103

$$y = mx - 2m - m^3$$

$$\Rightarrow \boxed{m^3 + (2-h)m + k = 0} \text{ ...} m_1, m_2, m_3$$

$$m_3 = \frac{-k}{\alpha} \Rightarrow k^2 = \alpha^2 h + (\alpha^3 - 2\alpha^2)$$

$$\Rightarrow \boxed{y^2 = \alpha^2 x + \alpha^2(\alpha - 2)}$$

For this to be part of $y^2 = 4x$

$$\boxed{\alpha = 2}$$

Q.104

$$I_n = \int_0^1 1\cdot(1 - x^{45})^n \, dx$$

$$\boxed{I_n = 45n(I_{n-1} - I_n)} \rightarrow \text{I.B.P}$$

$$\frac{I_{n-1}}{I_n} = \frac{45n+1}{45n}$$

$$\boxed{\frac{I_{44}}{I_{45}} = \frac{2026}{2025}}$$

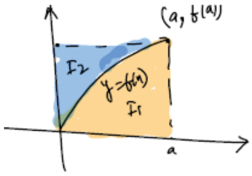
Q.105

$$|z - w|^2 + |z + w|^2 = 2(|z|^2 + |w|^2)$$

$$|\alpha_k| = 1 \quad \text{and} \quad |\alpha_{k+m}| = 1$$

$$\boxed{T_k = 4} \Rightarrow \frac{1}{4} \times (4 \times 2026) = \boxed{2026}$$

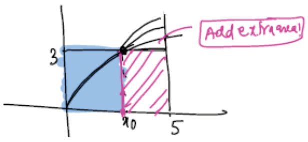
Q.106



$$I_1 = \int_0^a f(x)dx$$

$$I_2 = \int_0^{f(a)} f^{-1}(x)dx$$

$$I_1 + I_2 = af(a)$$



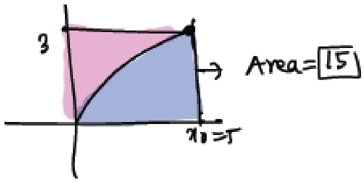
If f is surjective then $\exists\, x_0 \in [0,\infty)$ such that $f(x_0) = 3$ (let $x_0 < 5$) $\rightarrow$
That is what is discussed

If $x_0 < 5 \Rightarrow \underbrace{\int_0^{x_0} f(x) + \int_0^{3} f^{-1}(x)dx}_{\text{Fixed}} + \int_{x_0}^{5} f(x)dx$

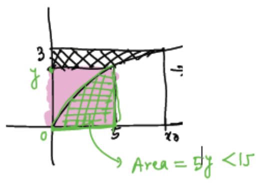
$\geq 3x_0 + (5 - x_0)3$

$\geq \boxed{15}$

For $\boxed{x_0 = 15}$



For $x_0 > 5$



$$\Rightarrow \left[\int_0^5 f(x)dx + \int_0^y f^{-1}(x)dx\right] + \int_y^3 f^{-1}(x)dx$$

$$= 5y + (3-y)5 + \delta = 15 + \delta \geq 15$$

Q.107

$$\cos\left(\frac{\pi}{2}-14x\right)=\cos 68x$$

$$\Rightarrow 68x = 2n\pi \pm\left(\frac{\pi}{2}-14x\right)$$

$$\Rightarrow 82x=\left(2n+\frac{1}{2}\right)\pi \quad \text{or } 54x=\left(2n-\frac{1}{2}\right)\pi$$

$$x=\frac{(4n+1)}{2}\frac{\pi}{82}$$

$$x\in[0,2\pi]$$

$$\Rightarrow 0\le 4n+1\le 328$$

$$\Rightarrow -\frac{1}{4}\le n\le 81.\text{something}$$

$$\downarrow n=0,1,2,\ldots,81$$

$$x=\frac{(4n-1)\pi}{108}$$

$$\Rightarrow \frac{1}{4}<4n\le 217$$

$$\Rightarrow n=1,2,3,\ldots,54$$

Number of overlapping values

$$\frac{(4i+1)}{2}\frac{\pi}{82}=\frac{(4j-1)}{2}\frac{\pi}{54}$$

$$\Rightarrow \frac{4i+1}{4j-1}=\frac{41}{27}\rightarrow\begin{cases} i=10 \\ j=7\end{cases}\Rightarrow \text{one pair}$$

Next choice is $\boxed{\frac{41}{27}\times\frac{3}{3}}$ or $\boxed{\frac{41}{27}\times\frac{9}{9}}$

$\boxed{\text{Two overlap!}}$

Q.108

$$\vec{N}=\overrightarrow{b_n}\times\overrightarrow{b_{n+1}}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & \sqrt{n}\\ 0 & 1 & \sqrt{n+1}\end{vmatrix}=(\sqrt{n+1}-\sqrt{n})\hat{i}$$

$$d_n=\frac{\left|(a_{n+1}-a_n)\cdot\vec{N}\right|}{|\vec{N}|}=\boxed{1}$$

Q.109

$\vec{X} = (a_1, a_2, \ldots, a_n)$

$\vec{Y} = (b_1, b_2, \ldots, b_n)$

$\vec{X}\cdot\vec{Y} = |\vec{X}||\vec{Y}|\cos\theta$ where $a_i = |f(i)|, b_i = \frac{1}{2^i}$

For equality case $\theta = 0 \Rightarrow |f(i)| = \frac{c}{2^i}$ for some $c > 0$

$1 = |f(10)| = \frac{c}{2^{10}} \Rightarrow C = 1024$

$\boxed{|f(i)| = \frac{2^{10}}{2^i} = 2^{10-i}}$

Since , f(1) =1

$f(1) = 1$

$|f(2)| = 2^8$

$\Rightarrow f(2) = \pm 2^8 \ldots$

So, "f(1) is fixed and Rest have 2 choices"

So, Total functions $= 2\times 2\times 2\times \ldots \times 2 = 2^9 = 512$

(Note: This is known as Cauchy-Schwarz inequality which is just an application of Dot product of vectors!)

Q.110

$\{2-d, 2, 2+d\} \rightarrow$ as sum $r_1 + r_2 + r_3 = 6$

$\downarrow$

d must be integer

If $d = 1, \beta = 6 > 0$

$d = 2, \beta = 0$ (Not allowed)

$d = 3, \beta = -10$

$d = 4, \beta = -24$

So, $\boxed{r_1 = -1, r_2 = 2, r_3 = 5}$

$\sum r^4 = \boxed{642}$

Q.111

Total number of subsets in F is $\binom{9}{3} = 84$.

In total sum each $w \in S$ appears in multiple subsets.
Let us fix one element w and chose 2 among 8.

$$\text{Frequency}\,(f) = \binom{8}{2} = 28.$$

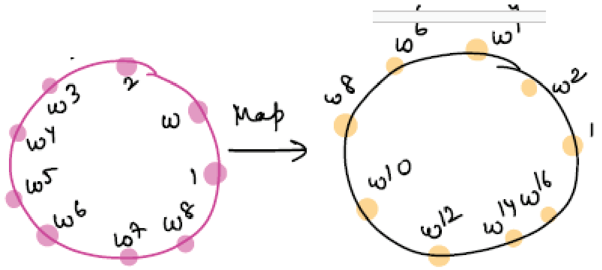
$$\text{So, Sum (asked)} = \sum_{z\in S} (\text{Frequency of } z)\cdot z^2 = 28\left(\sum_{z\in S} z^2\right)$$

$$= 28\left(\sum_{z\in S} z^2\right)$$

Since $\boxed{w_k = e^{\frac{i2\pi k}{9}}} \rightarrow$ are roots of $\boxed{z^9 - 1 = 0}$

$$\Rightarrow \boxed{\begin{matrix}\sum z_i = 0 \\ \sum z_i z_j = 0\end{matrix}} \Rightarrow \sum (z_i^2) = 0$$

Visual logic:



$\Rightarrow$ Automorphism

The nine root of unity maps to its square in same way (just as a permutation)

Q.112

$$\text{adj}^m(A) = \left(|A|^{\frac{(n-1)^m - (-1)^m}{n}}\right) \cdot A^{(-1)^m}$$

Where $\text{adj}^m(A) = \underbrace{\text{adj}(\text{adj}(\ldots(\text{adj}(A))}_{\text{m times}}$ (n is order of matrix)

Since B = adj A

$$\text{adj}^4(A) = \left(|A|^{\left(\frac{2^4-1}{3}\right)}\right) A = (|A|)^{\frac{15}{3}} A$$

$$= (8^{\frac{1}{15}})^{\frac{15}{3}} A$$

=2A

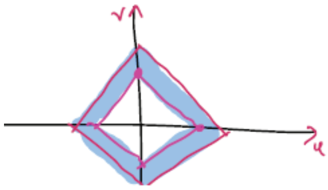
(To prove, use $A(\text{adj}A) = |A|I$)

M-2

Just take Determinant and get one-line sol

Q.113

$2 \le |u| + |v| \le 8$



$A(u, v) = 128 - 8 = 120$

$$\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

↑

Matrix as a Transformer (scales area!!)

$$A = \text{Area} = \frac{120}{\begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix}} = \frac{120}{|15 - 1|} = \frac{120}{14} = \frac{60}{7}$$

It gives idea (quick way to solve any such problem)

$1 \le |x + y| + |x - y| \le 2$ $\rightarrow$ Try this JEE in 2-lines!

$(x + y - 1)^2 + (2x + y + 1)^2 = 1$ $\rightarrow$ Try it as well!

Q.114

Tuplet's given by:

$(1,0,0,0,0)\xrightarrow{\text{Conjugate}}(0,-1,-1,-1,-1)\rightarrow$ Five position for 0

$\uparrow$

Five position for 1

List:

$\alpha = 1 \rightarrow (1,0,0,0,0)$ and $(0,-1,-1,-1,-1)$

$\alpha = 1 \rightarrow (1,0,0,0,0)$ and $(0,-1,-1,-1,-1)$

$\alpha = \omega^2 \rightarrow$

$\alpha = \omega^3 \rightarrow$ $\boxed{\text{Similarly}}$

$\alpha = \omega^4 \rightarrow$

$\boxed{C-2}$

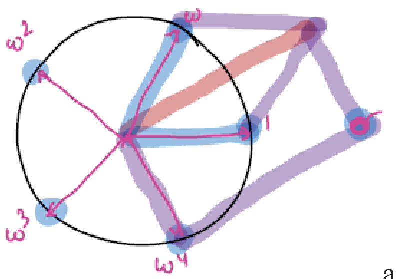
$\alpha = -1 \rightarrow (-1,0,0,0,0)$ and $(0,1,1,1,1)$

$\alpha = -\omega \rightarrow (0,-1,0,0,0)$ and $(1,0,1,1,1)$

$\alpha = -\omega^2 \rightarrow (0,0,-1,0,0)$ and $(1,1,0,1,1)$

$\alpha = -\omega^3 \rightarrow (0,0,0,-1,0)$ and $(1,1,1,0,1)$

$\alpha = -\omega^4 \rightarrow (0,0,0,0,-1)$ and $(1,1,1,1,0)$



always out

Q.115

Let $u^3=x$ so,

$$\int_0^1 \sqrt{x + x^{1/3}}\,dx = 3\int_0^1 u^2\sqrt{u^3 + u}\,du$$

$$I = \int_0^1 (3x^2+1)\sqrt{x+x^3}\,dx = \left[\frac{2}{3}(x+x^3)^{3/2}\right]_0^1 = \frac{4\sqrt{2}}{3}$$

Q.116

For each n:

$$\lim_{r\to\infty}\left(1+\frac{\frac{1}{n(n+1)}}{r}\right)^r = e^{\frac{1}{n(n+1)}}$$

$$\Rightarrow L = \prod_{n=2}^{\infty} e^{\frac{1}{n(n+1)}}$$

$$= e^{\sum_{n=2}^{\infty}\frac{1}{n(n+1)}} = e^{1/2}$$

$$\boxed{\ln(e^{1/2}) = 1/2}$$

Q.117

$$I = \frac{\pi}{2}\int_0^1 \frac{2}{\pi}\sin^{-1}\left(\sqrt{\frac{2}{\pi}\sin^{-1}(\sqrt{x})}\right)dx$$

Let $f(x) = \dfrac{2}{\pi}\sin^{-1}(\sqrt{x})$

$$I = \frac{\pi}{2}\int_0^1 f(f(x))dx \quad \left.\begin{matrix}\sin^{-1}\sqrt{1-x} = \cos^{-1}\sqrt{x}\\ \cos^{-1}\sqrt{1-x} = \sin^{-1}\sqrt{x}\end{matrix}\right\} x \in (0,1)$$

$$= \frac{\pi}{2}\int_0^1 f(f(1-x))dx \text{ Since, } f(1-x) = \frac{2}{\pi}\cos^{-1}\sqrt{x},$$

$$\boxed{f(x)+f(1-x)=1}$$

$$\Rightarrow \boxed{I = \frac{\pi}{4}}$$

Q.118

$$\ln|f_n(x)| = \sum_{k=1}^{n} \ln|\cos(kx)|$$

$$\Rightarrow \frac{f_{n'}(x)}{f_n(x)} = \sum_{k=1}^{n} -k\tan(kx)$$

$$\Rightarrow \frac{f_{n''}(x)f_n(x) - (f_{n'}(x))^2}{(f_n(x))^2} = \sum_{k=1}^{n} -k^2\sec^2(kx)$$

$$\Rightarrow |f_{n''}(0)| = \left|-\sum_{k=1}^{n} k^2\right| = \frac{n(n+1)(2n+1)}{6} > 2026$$

FOR

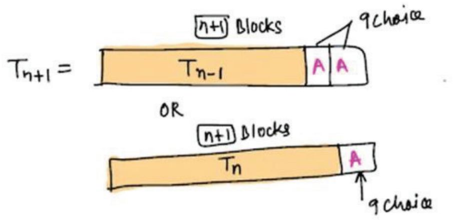
$$\boxed{\begin{array}{l} n = 17, \quad S = 1785 \\ n = 18, \quad S = 2109 \end{array}}$$

Hence n = 18

$$\boxed{\begin{array}{l} f\ (x) = \text{even } f^n \\ f\,'(x) = \text{odd } f^n \\ f\,''(x) = \text{even } f^n \\ f\,'''(x) = \text{odd } f^n \end{array}}$$

So, $\boxed{f_n\,'''(0) = 0 \quad \forall n \in \mathbb{N}}$

Q.119



$$\boxed{T_{n+1} = 9T_{n-1} + 9T_n}$$

$$\left|\frac{T_n + T_{n-1}}{T_{n+1}}\right| = 9 \ (T_1 = 9, T_2 = 90)$$

Q.120

JUST EXPAND AND USE VANDERMONDE'S YOUR DAY IT IS, IS NOT IT?

Q.121

$$\sum_{i=1}^{3}\frac{1}{1-\cos\theta_i} \text{ (Problem is we want multiple of 3 NOT 9)\}}$$

$\cos 3\theta_i = 1/2$ for all i.

$4\cos^3\theta_i - 3\cos\theta_i = 1/2$

$$x_i = \cos\theta_i \Rightarrow \sum_{i=1}^{3}\frac{1}{1-x_i}$$

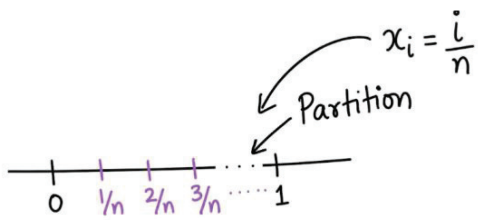
$f(x) = 4x^3 - 3x - 1/2 = 4(x-x_1)(x-x_2)(x-x_3)$

$$\frac{1}{1-x_1}+\frac{1}{1-x_2}+\frac{1}{1-x_3} = \boxed{18}$$

(use Transformation of roots!)

Question checks your observation!

Q.122



$f(y) = (1-y)\cos 4y$, Put $y_k = \frac{k}{n}$ for $k \in [1, n]$.

$$\sum \frac{n-k}{n^2}\cos\left(\frac{4k}{n}\right) = \sum_{k=1}^{n} f(y_k)(x_i - x_{i-1})$$

$$= \int_0^1 f(y)dy = \frac{1-\cos 4}{16}$$

$$= \frac{\sin^2 2}{8}$$

Q.123

Let $T = (2+\sqrt{3})^n$

$T_n' = (2-\sqrt{3})^n$

$T_n + T_n' = \text{even}$

$\Rightarrow \boxed{f_n + T_n' = 1}$

$$\frac{f_5^2}{1-f_5} = \frac{(1-T_{5'})^2}{T_{5'}} = \frac{\left(1-\frac{1}{T_5}\right)^2}{\frac{1}{T_5}} = \frac{(T_5-1)^2}{T_5}$$

$= T_5 + \frac{1}{T_5} - 2$

$= T_5 + T_5' - 2$

You can use Newton's Sum

$= (2+\sqrt{3})^5 + (2-\sqrt{3})^5 - 2$

$= 724 - 2 = \boxed{722}$

(Binomial theorem with quadratic flavour!)

Q. 124

$M = \{1, 2\}$

$P(M) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$.

Total collections S is

$|P(P(M))| = 16$.

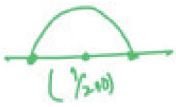
$\bigcup_{A \in S} A = M \Rightarrow \{1, 2\} \subseteq \cup$.

Valid collections = 16 - (4 + 4 - 2) = 10

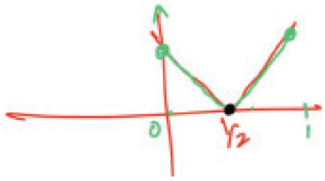
Q.125 Period of f and g (T) = 1

on [0, 1], $y=\sqrt{x-x^2}$

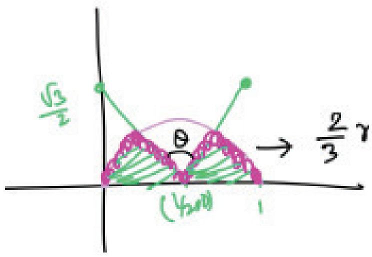
$\Rightarrow (x-1/2)^2+y^2=1/4$



second fn: on [0, 1] simplifies to $y=\sqrt{3}\left|\frac{1}{2}-x\right|$



So, for T = 1



$\theta=\pi/3$

$\rightarrow$ 2/3 rd of Semicircle

So, $\int_0^1 \min(f,g)dx=\frac{\pi}{24}\times 2$

Q.126

M-1: $g(x) = x^7 + 6x^5 + 3x^3 + 1$

$g'(x) \ge 0 \quad \forall\, x \in \mathbb{R} \quad$ (g is strictly increasing)

$f(g(x)) = 9x + 5$

$\Rightarrow f(x) = 9g^{-1}(x) + 5$

$\int_{-9}^{11} f(x)dx = \underbrace{9\left(\int_{-9}^{11} g^{-1}(x)dx\right)}_{1} + 100 = 100 \quad \left(\text{as } \int_{-9}^{11} g^{-1}(x)dx = 0\right)$

Since; $\int_{-9}^{11} g^{-1}(x)dx + \int_{-1}^{1} g(x)dx = 2$

Now, $\int_{-1}^{1} g(x)dx = \int_{-1}^{1} (x^7 + 6x^5 + 3x^3 + 1)dx$

$= \left[\frac{x^8}{8} + x^6 + \frac{3x^4}{4} + x\right]_{-1}^{1}$

$= \left(\frac{1}{8} + 1 + \frac{3}{4} + 1\right) - \left(\frac{1}{8} + 1 + \frac{3}{4} - 1\right)$

$= \left(\frac{1}{8} + \frac{3}{4} + 2\right) - \left(\frac{1}{8} + \frac{3}{4}\right)$

=2

Use result: $\boxed{\int_{a}^{b} f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx = b \cdot f(b) - a \cdot f(a)}$

Think Geometrically!

M-2

Let $x = u^7 + 6u^5 + 3u^3 + 1$

$f(x) = 9u + 5$

$dx = (7u^6 + 30u^4 + 9u^2)du$

$\int_{-9}^{11} f(x)dx = \int_{-1}^{1} (9u + 5)(7u^6 + 30u^4 + 9u^2)du$

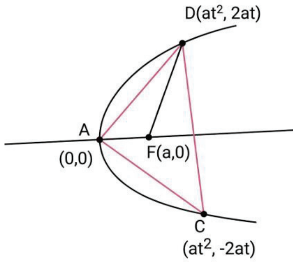
=100

Q.127

King's Rule! (Secretly inside!)

$2I = \int_{0}^{\pi/4} \frac{dx}{2^{1/4}} = \frac{\pi}{4 \times 2^{1/4}}$

Q.128



Let, $y^2 = 4ax$ (Shift to origin (0,0))

$$\frac{at^2 + at^2 + 0}{3} = a$$

$$\boxed{t^2 = 3/2}$$

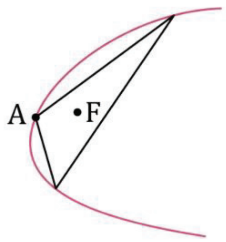
$$FB = a + at^2 = a\left(1 + \frac{3}{2}\right) = \frac{5a}{2} = 3$$

$$\boxed{a = \frac{6}{5}}$$

$$\text{Area} = \frac{1}{2} \times y \times at \times at^2 = 2a^2t^3 = 2 \times \frac{36}{25} \times \left(\frac{3}{2}\right)^{3/2}$$

$$= \frac{72}{25} \times \sqrt{\frac{27}{8}} = \frac{54\sqrt{6}}{25}$$

Why we took symmetric?



$y_1 + y_2 = 0$ (as A is (0,0)) This forces $y_1 = -y_2$ (Hence Symmetric!)

Bonus problem:

$$\int \frac{x^4 e^x}{(x^4 + 4x^3 + 12x^2 + 24x + 24 + 72e^x)^2} dx$$

$$= \int \frac{x^4 e^{-x}}{(e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + 72)^2} dx$$

$$= \frac{e^x}{x^4 + 4x^3 + 12x^2 + 24x + 24 + 72e^x} + C$$

Q.129

$$\cos^{-1}a-\cos^{-1}b=\cos^{-1}\left(ab+\sqrt{1-a^2}\sqrt{1-b^2}\right)$$

Roots of $k^4+2k^3+3k^2+2k+2=0 \leftarrow \begin{cases} i \\ -i \end{cases}$ (observe)

So factor $(k^2+1)((k+1)^2+1)$

So, $\cos^{-1}\left(\dfrac{1}{\sqrt{(k+1)^2+1}}\right)-\cos^{-1}\left(\dfrac{1}{\sqrt{k^2+1}}\right)$

$$=\cos^{-1}\left(\frac{1}{\sqrt{k^2+1}}\times\frac{1}{\sqrt{(k+1)^2+1}}+\sqrt{\left(1-\frac{1}{k^2+1}\right)}\cdot\sqrt{1-\frac{1}{(k+1)^2+1}}\right)$$

$$=\cos^{-1}\left(\frac{k^2+k+1}{\sqrt{k^4+2k^3+3k^2+2k+2}}\right)$$

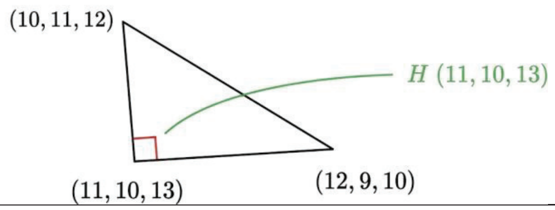
$$\sum \text{Series}=\cos^{-1}\left(\frac{42}{\sqrt{2\cdot 1682}}\right)=\cos^{-1}\left(\frac{21}{29}\right)$$

Q.130

$L_1 \cap L_2=(11,10,13)$

$L_2 \cap L_3=(12,9,10)$

$L_1 \cap L_3=(10,11,12)$



Q.131

$x^3+x+1=0 \rightarrow$ Roots x_1,x_2,x_3

$\Rightarrow x^3+x=-1$

$\Rightarrow x^6+x^2+2x^4=1$

$\Rightarrow y^3+y+2y^2=1$

$\Rightarrow y^3+2y^2+y-1=0 \rightarrow y_1,y_2,y_3$ where

$y_i=x_i^2\ \forall\, i=1,2,3$

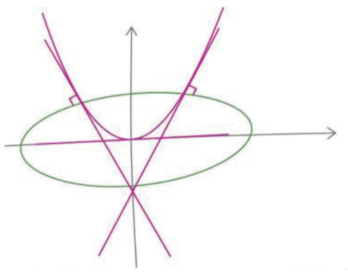
$\Rightarrow P(x)=\lambda(x^3+2x^2+x-1)$

$P(0)=-1 \Rightarrow \lambda=1$

$\Rightarrow \mathbf{P(9)=899}$

(Problem Tests your Decoding-Capability!)

Q.132



n(F) =6

$$ax\sec\theta - by\csc\theta = a^2 - b^2$$

$$y = \left(\frac{a}{b}\tan\theta\right)x - \frac{(a^2-b^2)\sin\theta}{b}$$

$$y = (8t)x - 4t^2 \quad \text{at } (t, 4t^2)$$

$$\Rightarrow \boxed{\frac{4a^2}{64b^2}\tan^2\theta = \frac{(a^2-b^2)\sin\theta}{b}}$$

$$\Rightarrow \sin\theta\left(\frac{a^2\sin\theta}{16b\cos^2\theta} - (a^2-b^2)\right) = 0$$

Case I

$\sin\theta = 0$ gives $\boxed{\theta = 0, \pi}$ $\rightarrow$ x-axis (one-line) $\rightarrow$ gives $t = 0$

Case II

$$\boxed{(a^2-b^2)\sin^2\theta + \frac{a^2}{16b}\sin\theta - (a^2-b^2) = 0}$$

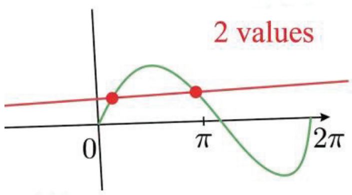
Let $u = \sin\theta$

$$f(u) = (a^2-b^2)u^2 + \frac{a^2}{16b}u - (a^2-b^2)$$

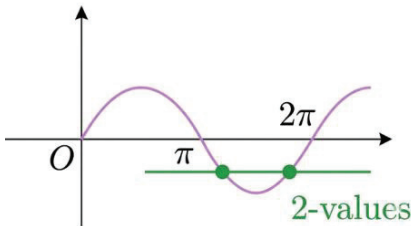
Since $a \neq b$, Product of roots = -1 roots are real. $\boxed{\alpha\beta = -1}$

So $\alpha \in (0,1]$ and other $\beta \in [-1, 0)$.

Now if $\sin\theta_1 = \alpha \Rightarrow \sin\theta_2 = \beta$ (No Solution as out of Domain)



$\underline{\text{OR}}$ $\sin\theta_1 = \beta \Rightarrow \sin\theta_2 = \alpha$ (out of Domain, No solution)

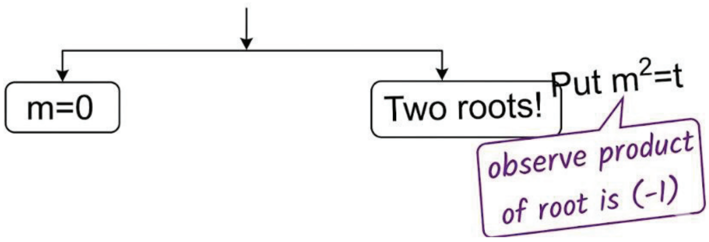


Total 3 solutions including –axis

$$y = mx + \frac{m^2(a^2 - b^2)}{a^2 + m^2b^2} \quad \text{(NOT TO REMEMBER!!)}$$

$$y = mx - \frac{m^2}{16}$$

$$\Rightarrow m^2\left(\frac{m^2}{256} - \frac{(a^2 - b^2)^2}{a^2 + b^2m^2}\right) = 0$$



$\boxed{\alpha\beta = -1} \Rightarrow \alpha > 0, \beta < 0 \quad m^2 = \alpha \Rightarrow 2$ roots

Consider one more example:

$$S = \left\{(l, m) \in \mathbb{N} \times \mathbb{N} : lx + my + 1 \text{ is tangent to parabola } y^2 = 4x\right\}$$

$$T = \left\{(l, m) \in \mathbb{Z} \times \mathbb{N} : lx + my + 1 \text{ is Normal to Ellipse } \frac{x^2}{4} + \frac{y^2}{1} = 1\right\}$$

Then $|n(S) + n(T)|$ (Think!)

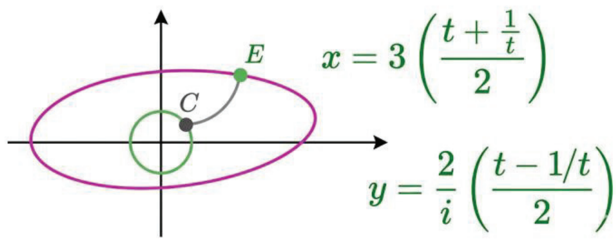
$$2x\sec\theta - y\csc\theta = 3$$

$$-lx - my = 1$$

$$\Rightarrow \boxed{\frac{4}{9l^2} + \frac{1}{9m^2} = 1} \quad \text{after elimination}$$

$|n(S)| \to \infty, \quad |n(T)| = 0$

Q.133



where $t = e^{i\theta}$ (Map)

Let Circumcircle $(x-h)^2 + (y-k)^2 = R^2$. Put this value we get:

$$5t^4 - 4(3h - 2ik)t^3 + (\ldots)t^2 = \cdots$$

$$t_1 \quad t_2 \quad t_3 \quad \underbrace{t_4}_{\text{Extraneous}}$$

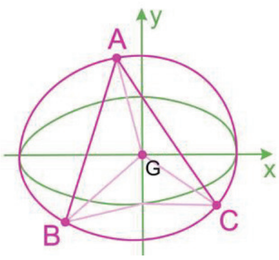
$t_1 + t_2 + t_3 = 0$ (centroid is origin). $\Rightarrow \boxed{t_4 = \frac{4(3h-2ik)}{5}}$

Since $t = e^{i\theta}, |t_4| = 1$ (for all t_1, t_2, t_3, t_4) gives $\boxed{144x^2 + 64y^2 = 25}$

Logic 2

$\sum \cos\alpha = 0$ and $\sum \sin\alpha = 0 \Rightarrow \boxed{\cos(\alpha - \beta) = -1/2}$

$\Rightarrow |\alpha - \beta| = \frac{2\pi}{3}$ etc.

Geometrically: 

(ABC must be equilateral)

$|\alpha - \beta| = |\beta - \gamma| = |\gamma - \alpha| = \frac{2\pi}{3}$ (Roots of unity!)

Q.134

$$g'(x) = \frac{\sin(\pi\sqrt{x^2})}{x^2} \cdot 2x = \frac{2\sin(\pi x)}{x}$$

$$g'(x) = 0 \Rightarrow x_1 = 1, x_2 = 2, x_3 = 3, \ldots$$

$$\text{Ag : Let } h(x) = g'(x) = \frac{2}{x}\sin(\pi x)$$

$$h'(x) = 0 \Rightarrow 2\left[\frac{\pi x\cos(\pi x) - \sin(\pi x)}{x^2}\right] = 0$$

$$\Rightarrow \tan(\pi x) = \pi x$$

$$n(A_g) = 2 \quad (\text{from } \tan\theta = \theta)$$

$$\boxed{\theta \in (\pi, 3\pi/2), \quad \theta \in (2\pi, 3\pi)}$$

$$\text{Bg} : h''(x) = \frac{2}{x^3}\left[(2 - \pi^2 x^2)\sin\pi x - 2\pi x\cos\pi x\right]$$

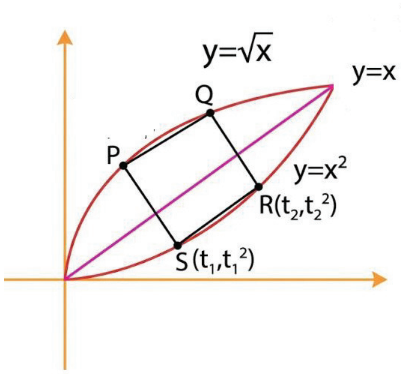
$$\Rightarrow h''(x) = 0 \text{ leads to}$$

$$\boxed{\tan\pi x = \frac{2\pi x}{\pi^2 x^2 - 2}}$$

$$\boxed{\tan\theta = \frac{2\theta}{\theta^2 - 2}} \rightarrow \text{Graph :}$$

$$n(B_g) = 2$$

Q.135



$$\frac{t_2^2-t_1^2}{t_2-t_1}=1 \Rightarrow \boxed{t_1+t_2=1}$$

$$t_1=t,\ t_2=1-t$$

$$PS=(t^2-t)^2+(t^2-t)^2=2(t-t^2)^2$$

$$SR=2(2t-1)^2$$

$$\Rightarrow 2(2t-1)^2=2(t-t^2)^2$$

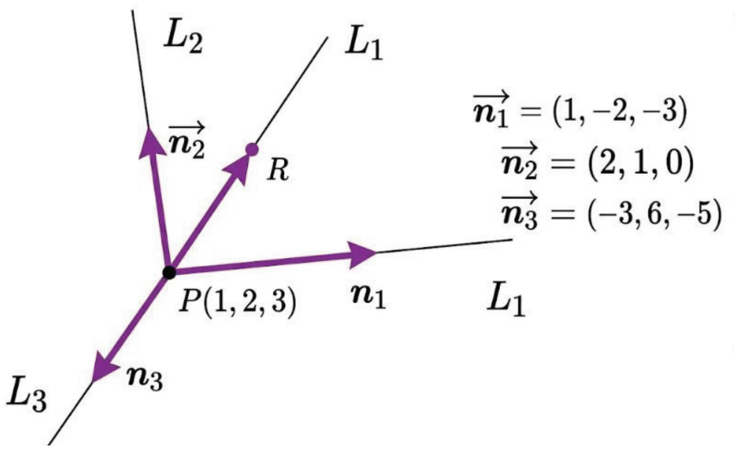
$$\Rightarrow t\boxed{^2+t-1=0}$$

$$t=\frac{\sqrt{5}-1}{2}$$

$$\text{Area } A=2(2t-1)^2=2(\sqrt{5}-2)^2$$

$$=18-8\sqrt{5}$$

Q.136



$\overrightarrow{PR} = x\hat{n}_1 + y\hat{m}_2 + z\hat{n}_3$ (as orthogonal)

$PA = PB = PC = \lambda = 5$

$\Rightarrow \overrightarrow{PA} = \lambda\hat{n}_1, \overrightarrow{PB} = \lambda\hat{n}_2, \overrightarrow{PC} = \lambda\hat{n}_3$

Let variable line have a unit vector $\hat{u}$ (direction).

Reflection of PA in $\hat{u}$ is given by

$\overrightarrow{PA_1} = 2(\overrightarrow{PA}\cdot\hat{u})\hat{u} - \overrightarrow{PA} = 2\lambda(\hat{n}_1\cdot\hat{u})\hat{u} - \lambda\hat{n}_1$

plane P_1 passes through A_1 and is perpendicular to L_1.

$(\vec{R} - \overrightarrow{PA_1})\cdot\hat{n}_1 = 0$

$\boxed{\vec{R}\cdot\hat{n}_1 = x}$

$\Rightarrow \boxed{x = \lambda(2(\hat{n}_1\cdot\hat{u})^2 - 1)}$

$y = \lambda(2(\hat{n}_2\cdot\hat{u})^2 - 1)$

$z = \lambda(2(\hat{n}_3\cdot\hat{u})^2 - 1)$

$x + y + z = 2\lambda\left((\hat{n}_1\cdot\hat{u})^2 + (\hat{n}_2\cdot\hat{u})^2 + (\hat{n}_3\cdot\hat{u})^2 - 3\right)$

$\Rightarrow \boxed{x + y + z = -\lambda}$

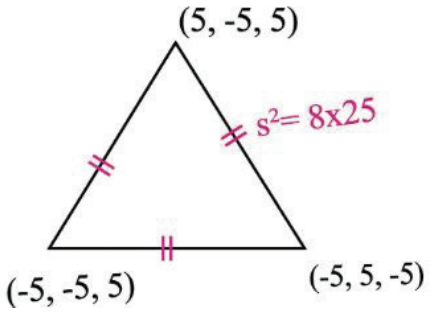
Since, $\boxed{\sum(\hat{n}_i \cdot \hat{u})^2 = 1}$

Since $(\hat{n}_i \cdot \hat{u})$ is ranged to [0, 1]

$\Rightarrow x, y, z \in [-\lambda, \lambda]$

So, locus is region x + y + z = -5 bounded by cube $[-5, 5]^3$.

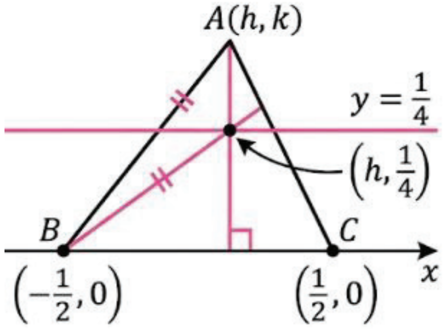
This is equilateral triangle:



Area $= \dfrac{\sqrt{3}}{4} \times 8 \times 25 = 50\sqrt{3}$

$\boxed{A = 50\sqrt{3}}$

Q.137



$BH \perp AC$

$$\Rightarrow \left(\frac{1/4}{h+1/2}\right)\times\left(\frac{k}{h-1/2}\right)=-1$$

$\Rightarrow \boxed{\mathbf{y=1-4x^2}} \rightarrow$ Locus is parabolic segment!

$S = 1/2 \times 1 \times y = y/2$

Since, at x=0, vertex A is highest, $y_{max} = 1$ (Isosceles triangle case)

For an acute triangle H must be strictly inside the triangle, hence $y_A > y_H$ (vertex A must be higher than H)

As $y_A \rightarrow 1/4, \angle A \rightarrow \pi/2 \Rightarrow \mathbf{y > 1/4}$

So, $\text{area}(S) \in (1/8, 1/2]$

$$\boxed{\mathbf{\frac{1}{m}+\frac{1}{M}=10}}$$

(Think of same problem with incenter (I), Centroid (G), Circumcenter (O) as well!

check solution at " COORDINATE PLAYLIST")

Q.138 $(\log_{\cos x} \sin^2 x)(\log_{\cos^2 x} \sin x) = 4$

$\text{Domain} : \cos x \in (0,1) \text{ and } \sin x > 0$

$x \in \bigcup_{m\in\mathbb{Z}} (2m\pi, 2m\pi + \frac{\pi}{2}) \rightarrow$ First Quadrant Restriction!

$\boxed{\log_{\cos x} \sin x = 2}$

$\Rightarrow \sin x = \cos^2 x = 1 - \sin^2 x$

$\sin x = \dfrac{-1+\sqrt{5}}{2}$

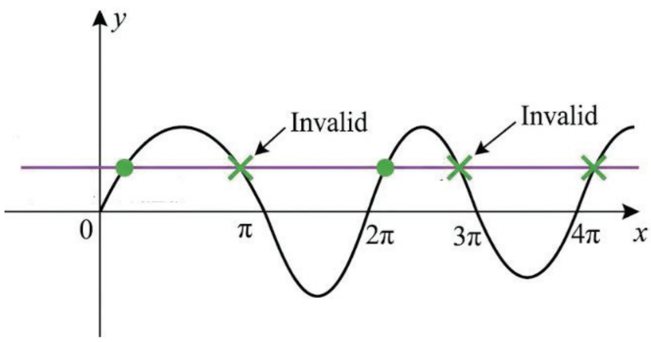
$\boxed{\log_{\cos x} \sin x = -2} \Rightarrow \sin x \cos^2 x = 1$

Since, $\sin x \in (0,1)$ and $\cos^2 x \in (0,1)$

Hence, $\sin x \cos^2 x \neq 1$.

$\boxed{\sin x = \dfrac{\sqrt{5}-1}{2} = \alpha > 0}$

with Domain restriction gives $k \in \{9,10\}$



Q. 139

$$A=\begin{bmatrix} s_2 & 3s_3 & s_6 \\ 3s_1 & 3s_5 & s_1 \\ 2s_0 & s_2 & 4s_3 \end{bmatrix}, \quad B=\begin{bmatrix} 2s_2 & 3s_3 & 4s_6 \\ 4s_1 & 4s_5 & 2s_1 \\ 3s_0 & 2s_2 & 4s_3 \end{bmatrix}$$

Observe $A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}, B\begin{bmatrix}1\\1\\1\end{bmatrix}=2\begin{bmatrix}1\\1\\1\end{bmatrix}$

Let $e=\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\boxed{Ae=e\Rightarrow A^{-1}e=e}$

$\boxed{Be=2e\Rightarrow 2B^{-1}e=e}$

$(AB)^2e=ABA(Be)=AB(2e)=4e$

$\Rightarrow \text{Row Sum of } (AB)^3=(\text{Extend above!})$

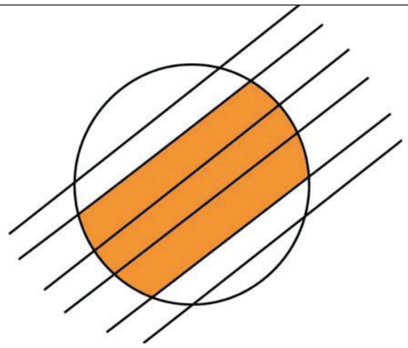
Now, $(A^{-1}B)e=A^{-1}(Be)=A^{-1}(2e)=2(A^{-1}e)=2e$

$\boxed{(A^{-1}B)^2e=4e}$

Row Sum of B^{-1} are $\frac{1}{2}$ and row sum of $A^{-1}B^{-1}$ are $\frac{1}{2}$.

$|\text{Total Sum}|=|3\times 8|=\boxed{24}$

Q.140



Picture speaks it all

Q.141

$$|z^2+1|^2 = |z+1|^4$$

$$\Rightarrow |z|^4 + 4x^2 - 2|z|^2 + 1 = (|z|^2 + 2x + 1)^2$$

Let $|z|^2 = t$

$$\Rightarrow \boxed{-4t = 4xt + 4x}$$

$$\Rightarrow t = \frac{-x}{1+x} \quad \text{since } t \geq 0$$

$$\frac{x}{1+x} \leq 0 \Rightarrow x \in (-1, 0]$$

$$\boxed{\text{So, x must be 0 (as } x \in \mathbb{Z})}$$

And $x = 0, t = 0 \Rightarrow \boxed{y = 0}$

$$\boxed{S = \{0\}} \quad n(S) = 1, \quad \sum_{Z \in S} |z|^2 = 0$$

Q. 142

$$A : \{\square, 6, \square, \square, \square, \square, 10, \square, \square\} \to (10 + 2d)$$

$$\bar{A} : \{\square, 10, \square, \square, \square, \square, 6, \square, \square\} \to (6 - 2d)$$

$$\left|\sum_{A \in S} f(A)\right| = \binom{9}{2} \times 16$$

Q.143

$|M| \neq 0$ for invertible

$n_0 + n_1 = 9$

Expression $= |n_1 - (9 - n_1)| = |2n_1 - 9|$

If $n_1 = 9, M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, |M| = 0$ (Rejected)

$n_1 = 8$ (Not possible as two rows must be 1)

$n_1 = 7,\ M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, |M| \neq 0$

To minimize $|2n_1 - 9|$, n_1 should be close to 4 or 5.

$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, |M| \neq 0$

min value = |2. 5 - 9| = 1

$$\boxed{\begin{array}{l} \max_{M \in S} |n_1(M) - n_0(M)| = 5 \\ \min_{M \in S} |n_1(M) - n_0(M)| = 1 \end{array}}$$

Q.144

$$K_n = \int_0^\infty \frac{dx}{\left(e^{x/2} + e^{-x/2}\right)^{2n}}$$

$$K_{13} = \int_0^\infty \frac{e^{13x}\, dx}{(e^x + 1)^{26}}$$

Let $t = \dfrac{e^x}{1+e^x}$ (motivation 😍)

$$dt = \frac{e^x\, dx}{(1+e^x)^2}$$

$$K_{13} = \int_{1/2}^{1} t^{12}(1-t)^{12}\, dt$$

$$= \frac{1}{2}\int_0^1 t^{12}(1-t)^{12}\, dt (\text{King'sRule})$$

$$\boxed{K_{13} = \frac{1}{2}\beta(13,13)}$$

$$\boxed{\lambda + \alpha = 15}$$

Q.145

$$P_n(x) = \sum_{k=0}^{n} x^k \left(\sum_{j=0}^{n-k} (x+1)^j (x-1)^{n-k-j}\right)$$

$$= \sum_{k=0}^{n} x^k \left(\frac{(x+1)^{n-k+1} - (x-1)^{n-k+1}}{2}\right)$$

$$= \frac{1}{2}\left((x+1)^{n+2} + (x-1)^{n+2} - 2x^{n+2}\right)$$

Coefficient of x^{2k}in$P_{2m}(x)$ is $\dbinom{2m+2}{2k}$.

So, Coefficient of x^4 in $P_6(x)$ is $\dbinom{8}{4} = 70$.

Q. 146

$$\frac{18r}{9r^4-15r^2+10}=\frac{18r}{9+(9r^4-15r^2+1)}$$

$$=\frac{2r}{1+\frac{(9r^4-15r^2+1)}{9}}$$

$$\frac{9r^4-15r^2+1}{9}=\frac{(3r^2)^2+1^2-6r^2-9r^2}{9}$$

$$=\frac{(3r^2-1)^2-9r^2}{9}$$

$$=\frac{(3r^2+3r-1)(3r^2-3r-1)}{9}$$

$$\tan^{-1}\left(\frac{2r}{1+\underbrace{\left(\frac{3r^2-3r-1}{3}\right)}_{A_{r-1}}\underbrace{\left(\frac{3r^2+3r-1}{3}\right)}_{A_r}}\right)$$

$$=\tan^{-1}\left(\frac{A_r-A_{r-1}}{1+A_rA_{r-1}}\right)$$

$$\text{Sum}=\tan^{-1}\frac{3}{17}$$

Line of Attack:

FOR ALL JEE PROBLEMS!

$$\tan^{-1}\left(\boxed{\#}\right)\downarrow\frac{x-y}{1+xy}$$

put (1 + 😍)
OR (Divide to get 1 somehow)

$$\text{OR}\quad \underbrace{\text{Something}}_{\substack{\downarrow\\ \text{Dividebythis!}}}+\text{Rest})$$

Q.147

$$\frac{x^2}{26^2}+\frac{y^2}{20^2}=1$$

$$\boxed{S=a(1+e)}\rightarrow \text{Semiperimeter}$$

$$r=\frac{\Delta}{s}\Rightarrow r(p)=\frac{\text{Area}(PF_1F_2)}{a(1+e)}$$

$$=\frac{\frac{1}{2}\times 2ae\times b}{a(1+e)}$$

$$\boxed{r_{max}=\frac{be}{1+e}}$$

BONUS:

Consider $E:\dfrac{x^2}{26^2}+\dfrac{y^2}{20^2}=1.$

A normal drawn at a variable point P on E. Let d be the perpendicular distance from center of ellipse to this normal. If the maximum value of d is d_{max}. The value of d_{max} is

$$ax\sec\theta-by\csc\theta=a^2-b^2$$

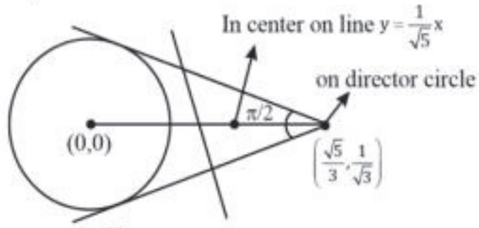
$$d=\frac{|a^2-b^2|}{\sqrt{a^2\sec^2\theta+b^2\csc^2\theta}}$$

$$d_{max}=a-b=26-20=6$$

Q.148

Ans. 2

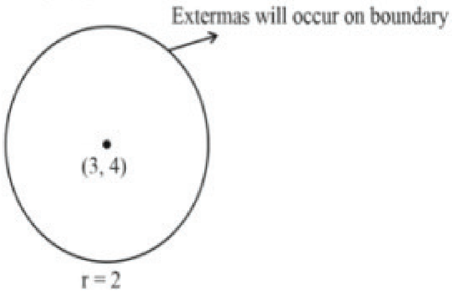
$SS_1=T^2\rightarrow$ Observation



So, $\alpha=\dfrac{\sqrt{5}}{\sqrt{3}},\beta=\dfrac{1}{\sqrt{3}}$ and $\dfrac{a}{b}=\sqrt{5}$

Q.149

Ans. 10



$x = 3 + 2\cos\theta$

$y = 4 + 2\sin\theta$

$$\frac{|2x-5y|}{7} = \frac{16+(24\cos\theta - 10\sin\theta)}{7}$$

$$= \frac{16+26}{7} = 6$$

$$\frac{\sqrt{3}y}{|x-3|} + 1 = \frac{\sqrt{3}(4+2\sin\theta)}{2|\cos\theta|} + 1$$

$$\text{Let } f(\theta) = \frac{4+2\sin\theta}{2\cos\theta} = \frac{2+\sin\theta}{\cos\theta} (\cos\theta > 0)$$

$f'(\theta) = 2\sec\theta\tan\theta + \sec^2\theta = 0$

$\sec\theta = 0$ of $2\tan\theta = -\sec\theta$

$$\Rightarrow \boxed{\sin\theta = \frac{-1}{2}}$$

similarly for $\cos\theta < 0$

$$\min\left(\frac{\sqrt{3}y}{|x-3|} + 1\right) = \left(\frac{\sqrt{3}(y-1)}{2\times\frac{\sqrt{3}}{2}} + 1\right) = 4$$

Q.150

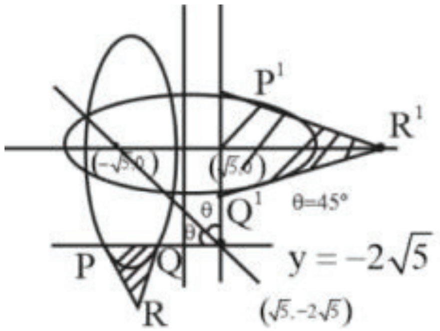
Ans. 8

By A.M – G.M

$$\sin^2\theta = \cos^2\theta \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\sum_{\theta\in S} \frac{2}{\sin\left(\frac{\pi}{2} + 4\theta\right)} = -8$$

Q.151 **Ans. 5**



$$\Delta = \frac{1}{2}\times\frac{8}{3}\times\left(\frac{9}{\sqrt{5}}-\sqrt{5}\right)=\frac{4}{3}\times\frac{4}{\sqrt{5}}=\frac{16}{3\sqrt{5}}$$

Q.152

$$f'(x)=3x^2-18x+24$$

$$f''(x)=6x-18 \Rightarrow x=3$$

f(3) = 8 (pt of inflection is (3, 8))

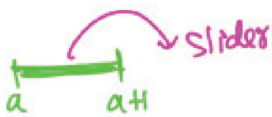
$\boxed{f(x+h)+f(h-x)=2k} \rightarrow$ about inflection

$$f(3+h)+f(3-x)=16$$

$\boxed{f(\alpha)+f(\beta)=16} \rightarrow$ Given

$$\frac{\alpha+\beta}{2}=3 \Rightarrow \boxed{\alpha+\beta=6}$$

Q.153 Find period



= All possible values of f(x)

Q.154

7 can cover either second or third place bijection. If it is in first place S would be 49, if fourth or later S is $3\times 6+4\times 7=46$

If f(2) = 7, f(1) must be 5. Hence 5! = 120 ways

If f(3) = 7, both max f(1) and max(f(1), f(2)) must be 6 which fores f(1) = 6.

Remaining has 6! = 120 ways.

So total 240 ways

Q.155

$n\rightarrow$ f(n) we have f(f(n)) = 3f(n)

Taking f on both side f(f(f(n))) = f(3n)

$$\Rightarrow f(3n)=3f(n)\forall n\in\mathbb{N}$$

n=1 gives f(3) = 3f(1), If f(1) = 1 then f(f(1)) = 1 and f(f(1)) = 3 which is contradict.

So $f(1)\neq 1$. Thus 3 = f(f(1)) > f(1) > 1

So f(1) = 2 and f(2) = f(f(1)) = 3.

Q.156

$$\alpha=\sum_{r=0}^{n}\frac{a_r^{\ 2}}{r+1}\quad,\quad \beta=\sum_{r=0}^{n-1}\frac{a_r a_{r+1}}{r+2}$$

$$\alpha=\sum\frac{\binom{n}{k}\binom{n}{k}\frac{n+1}{n+1}}{k+1}=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n}{n-k}$$

$$=\frac{\binom{2n+1}{n+1}}{n+1}$$

$$\beta=\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n}{n-k}\binom{n+1}{k+2}=\frac{1}{n+1}\binom{2n+1}{n+2}$$

$$\frac{\beta}{\alpha}=\frac{n}{n+2}=\frac{5}{6}\Rightarrow \boxed{n=10}$$

Q.157

$a_k=10$ for some $k\in\{1,2,3\}$

$a_n=a_1r^{n-1}, r\in\{2,3,4,\ldots\}$

$\boxed{a_1=10}$ $f(G)=10(1+r+r^2)$

$\boxed{a_2=10}$, $a_1r=10\Rightarrow r\,|\,10$ since $r>1, r\in\{2,5,10\}$

$a_3=10$, $a_1r^2=10\Rightarrow r^2\,|\,10$, No integer $r>1$

From $a_1=10$, $\min(f)=10(1+2+4)=70$

$a_2=10: r=2\Rightarrow a_1=5$ So, $f=5(1+2+4)=35$

m=35 with $r\le 10$ from $a_1=10, r=10, f=10(1+10+10^2)=1110$

From $a_2=10, r=10, a_1=1,$

$f=1(1+10+100)=111$

This M=1110

$$\boxed{\frac{m+M}{5}=229}$$

Q.158

$S_1 : (x-3)^2 + (y-4)^2 = 1$

$S_1 = \{2+4i, 4+4i, 3+3i, 3+5i\}$

$S_1 = \{2+4i, 4+4i, 3+3i, 3+5i\}$

$S_2 = \{5, -5, 4i, -4i\}$

$|z_1 - z_2|^2 = \{4, 10, 17, \ldots\}$

$\boxed{\min(|z_1 - z_2|) = 4}$

Q.159

$$\int_0^{\sqrt{2}} \left(\frac{\left(\frac{2}{x^2} - 1\right)}{\left(x + \frac{2}{x}\right)\sqrt{\left(x + \frac{2}{x}\right)^2 - 4}} \right) dx$$

Let $x + \frac{2}{x} = t$

$dt = \left(1 - \frac{2}{x^2}\right) dx$

$\boxed{I = 3}$

Q.160

$\Delta = 0$

$$\Rightarrow \begin{vmatrix} 13 & b & c \\ a-13 & 23-b & 0 \\ a-13 & 0 & 42-c \end{vmatrix} = 0$$

$$\Rightarrow \frac{13}{a-13} + \frac{b}{b-23} + \frac{c}{c-42} = 0$$

$$\Rightarrow \frac{13}{a-13} + \frac{b}{b-23} + 1 + \frac{c}{c-42} + 1 = 2$$

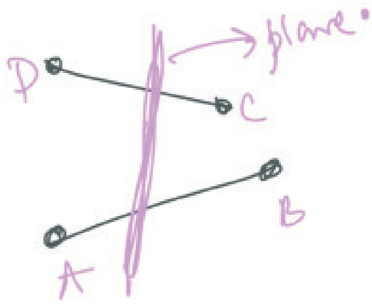
$$\Rightarrow \frac{13}{a-13} + \frac{23}{b-23} + \frac{42}{c-42} = 2$$

Q.161

plane in b/w D and ΔABC. $\binom{4}{1}=4$

2-2 partition

parallel to opposite edges mid point (3 pairs as T contains exactly 3 pairs of opposite edges



Q.162

$|A|=\pm 1 \quad \left(A^{-1} \text{ has integer entry, } |A^{-1}|=\frac{1}{|A|}\right)$

$\pm 1=|A|=a-b \Rightarrow a-b=1 \text{ or } a-b=-1$

Total Cases $\boxed{101+101=202}$

Q.163

$x^6-2x^5+x^3+x^2-x-1=(x^2-x)(x^4-x^3-x^2-1)+2x^2-2x-1$

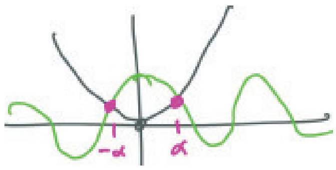
$S=2(a^2+b^2+c^2+d^2)-2(a+b+c+d)-4=0$

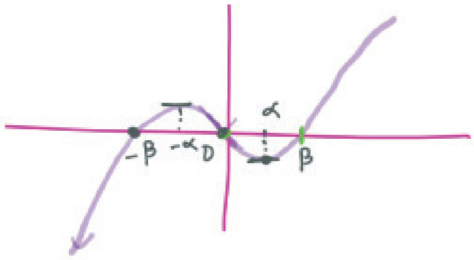
Q.164

$$S_0 = 0 \text{ as } \frac{x^4}{12} + \cos x + 1 > 0 \quad (-1 \le \cos x \le 1)$$

$$f'(x) = \frac{x^3}{3} - \sin x = g(x)$$

$$g'(x) = x^2 - \cos x$$





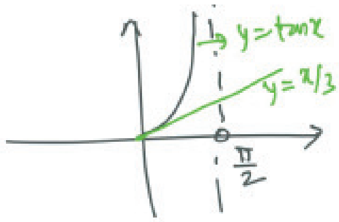
$$S_1 = \{0, \pm\beta\}$$

$$\alpha^2 = \cos\alpha$$

$$g(\alpha) = \frac{\alpha^3}{3} - \sin\alpha$$

$$g(\alpha) = \frac{\alpha\cos\alpha}{3} - \sin\alpha$$

$$= \cos\alpha\left(\frac{\alpha}{3} - \tan\alpha\right)$$



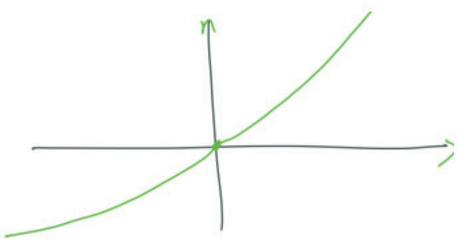
$$f''(x) = x^2 - \cos x$$

$$S_2 = \{\pm\alpha\}$$

$$f'''(x) = 2x + \sin x = g(n)$$

$$g'(x) = 2 + \cos x > 0$$

$$S_3 = \{0\}$$



$$\boxed{\begin{array}{l}\alpha \in (0, \pi/2) \\ \Rightarrow g(\alpha) < 0\end{array}}$$

$$f^4(x) = 2 + \cos x, \quad S_4 = \phi$$

$$f^5(x) = -\sin x, \quad S_5 = \{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\}$$

K=5

Q.165

$\boxed{\frac{21}{4}}$

$\sqrt{3} = 2 + 2\cos\theta \Rightarrow \boxed{\theta = \frac{\pi}{3}}$

$|\hat{a} \times \hat{b}| = \frac{\sqrt{3}}{2}$

let $\vec{v} = \hat{a} \times \hat{b}$, wewant $P = |\vec{v} \times \vec{c}|$

$\vec{v} \times \vec{c} = \vec{v} \times (\vec{a} + 2\vec{b} - 3\vec{v})$

$\boxed{P^2 = \frac{21}{4}}$ as $4|\hat{b}|^2 + \frac{25}{4}|\hat{a}|^2 - 2 \cdot 2 \cdot \frac{5}{2}(\hat{b} \cdot \hat{a})$

Q.166

$\frac{11}{10}$

$$\frac{(n^4 + 4) + (3n^2 + 10n + 6)}{2^n (n^4 + 4)}$$

$$= \frac{1}{2^n} + \frac{4n^2 + 8n + 8 - n^2 + 2n - 2}{2^n (n^2 + 2n + 2)(n^2 - 2n + 2)}$$

$$= \frac{1}{2^n} + \frac{1}{2^{n-2}(n^2 - 2n + 2)} - \frac{1}{2^n (n^2 + 2n + 2)}$$

$$\boxed{\text{Sum} = \frac{1}{2} + \frac{6}{10} = \frac{11}{10}}$$

Q.167

$\boxed{0.75 \text{ to } 0.80}$

$f(x) = \frac{2}{\pi}\sin^{-1}(\sqrt{x})$

$I = \frac{\pi}{2}\int_0^1 f(f(x))dx$

$$\boxed{\begin{array}{l} \sin^{-1}\sqrt{1-x} = \cos^{-1}\sqrt{x} \\ \cos^{-1}\sqrt{1-x} = \sin^{-1}\sqrt{x} \quad , x \in [0,1] \end{array}}$$

$f(1-x) = \frac{2}{\pi}\cos^{-1}(\sqrt{x})$

so, $\boxed{f(x) + f(1-x) = 1}$

$I = \frac{\pi}{2}\int_0^1 f(f(x))dx \Rightarrow \boxed{I = \frac{\pi}{4}}$

Q.168

$$p(x=k)=\frac{1}{6}\left(\frac{5}{6}\right)^{k-1},k=1,2,3,\ldots$$

$$E(x)=\sum_{k=1}^{\infty}kp(x=k)=\boxed{6}$$

$$Var(x)=\sum k^2p(x=k)-(E(x))^2$$

$$=\boxed{30}$$

Alternate:

$$\sum_{k=1}^{\infty}x^{k-1}=f(x)=\frac{1}{1-x},|x|<1$$

$$f'(x)=\sum_{k=1}^{\infty}kx^{k-1}=\frac{1}{(1-x)^2}$$

$$\text{at } E(x)=\frac{1}{6}f'(5/6)=6$$

$$f''(x)=\sum_{k=2}^{\infty}k(k-1)x^{k-2}=\frac{2}{(1-x)^3}$$

$$xf''(x)=\sum_{k=2}^{\infty}k(k-1)x^{k-1}=\frac{2x}{(1-x)^3}$$

$$\sum_{k=1}^{\infty}k^2x^{k-1}=xf''(x)+f'(x)$$

$$=\frac{2x}{(1-x)^3}+\frac{1}{(1-x)^2}=\frac{1+x}{(1-x)^3}$$

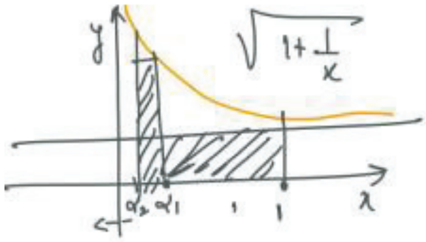
$$Var(x)=\frac{1}{6}g(5/6)-(E(x))^2=66-36=\ 30$$

Q.169

$$f(x)=p(x-1)^2(x-2)$$

$$f'(x)=p(x-1)(3x-5)=0\Rightarrow\ \boxed{c=\frac{5}{3}}$$

Q.170 $\sqrt{1+\frac{1}{x}} = n, \quad n \in Z^{+} \quad \boxed{x = \frac{1}{n^2-1}}$



$$\text{Sum} = 1\cdot(1-\alpha_1)+2(\alpha_1-\alpha_2)+3(\alpha_2-\alpha_3)+4(\alpha_3-\alpha_4)+\cdots$$

$$= 1+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\cdots\infty$$

$$= 1+\frac{1}{2^2-1}+\frac{1}{3^2-1}+\frac{1}{4^2-1}+\cdots$$

$$= 1+\sum_{n=2}^{\infty}\frac{1}{n^2-1}$$

$$= 1+\frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$$

$$= \frac{7}{4}$$

BONUS: Let $f(x) = \left[\log_2(1/x)\right]$ on $x \in (0,1]$

$$I_{m,n} = \int_0^1 [f(x)]^m x^n dx$$

$$I_{1,0} = \frac{1}{2}\sum n\left(\frac{1}{2}\right)^n = 1$$

$$I_{2,0} = \frac{1}{2}\sum n^2\left(\frac{1}{2}\right)^n = 3$$

$$I_{3,0} = \frac{1}{2}\sum n^3\left(\frac{1}{2}\right)^n = 13$$

$$I_{1,1} = \frac{3}{8}\sum n\left(\frac{1}{4}\right)^n = \frac{1}{6}$$

$$I_{1,2} = \frac{7}{24}\sum n\left(\frac{1}{8}\right)^n = \frac{1}{48}$$

Q.171

f(x) extracts 20th decimal digit of x.

For any x > 0,

$$x = N + \sum_{k=1}^{\infty} \frac{d_k}{10^k}, d_k \in \{0,1,\ldots,9\}, N \in \mathbb{Z}_{\geqslant 0}$$

$$10^{20} x = (10^{20} N + d_1 10^{19} + \ldots + d_{20}) + \sum_{k=21}^{\infty} \frac{d_k}{10^{k-20}}$$

$$[10^{20} x] = 10^{20} N + d_1 10^{19} + \ldots + d_{20}$$

$$10[10^{19} x] = 10(10^{19} N + d_1 10^{18} + \ldots + d_{19})$$

$f(x) = d_{20}$ (precisely 20th digit!)

Let $\beta = 5 - \sqrt{26}$

$$s_n = \alpha^n + \beta^n \in \mathbb{Z} \Rightarrow \alpha^{20} + \beta^{20} = k, k \in \mathbb{Z}$$

Since $\beta \in (-0.1, 0)$

$$0 < \beta^{20} < 10^{-20}$$

$$\alpha^{20} = (k-1) + \underbrace{(1-\beta^{20})}_{\downarrow}$$

$$0.\underbrace{9999\ldots9}_{20 \text{ times}} d_{21} d_{22} \ldots$$

$$\boxed{f(\alpha^{20}) = 9}$$

Q.172

$$\frac{dy}{dx} + \left(\frac{2 \cdot 2026 x^{2025}}{1 + x^{2026}}\right) y = \frac{2026 x^{2025}}{1 + x^{2026}}$$

(using Leibniz rule)

$$y(1 + x^{2026})^2 = \frac{(1 + x^{2026})^2}{2} + C$$

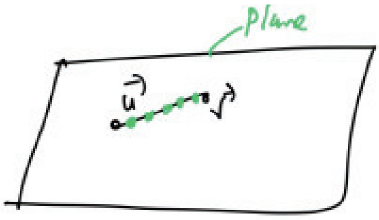
$$\boxed{y(0) = 0 \text{ gives } C = -\frac{1}{2}}$$

$$\boxed{\lim_{x \to \infty} y(x) = \frac{1}{2}}$$

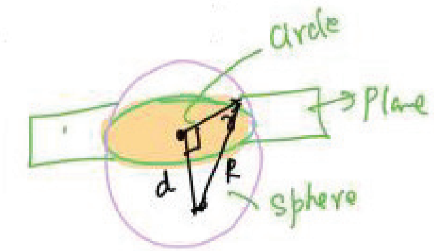
Q.173

$(1-t)\vec{u}+t\vec{v}\in S$ defines plane.

$x+2y+z=6$



$C: x^2+y^2+z^2=25$



$$r^2=R^2-d^2=19$$

Area of circle $=19\pi$

$A_{projection}=A\,|\cos\theta|$ where θ is angle between plane normal and z axis.

$$\begin{aligned} n &= (1,2,1) \\ \hat{k} &= (0,0,1) \end{aligned} \Rightarrow \cos\theta=\frac{n\cdot k}{|n||k|}=\frac{1}{\sqrt{6}}$$

$$\boxed{A_{proj.}=\frac{19\pi}{\sqrt{6}}}$$

Q.174

$$a_j = \binom{23}{j}\left(\frac{2}{5}\right)^j \Rightarrow \frac{a_j}{a_{j-1}} = \frac{23-j+1}{j}\cdot\frac{2}{5} = \frac{2(24-j)}{5j}$$

$\Rightarrow \dfrac{a_j}{a_{j-1}} \geq 1$ to be max $\Rightarrow j \leq 6.85$

Coefficient increases upto 6 and decreases.

So, $r = 6 \rightarrow$ Greatest coefficient index

Numerically greatest coefficient index (k)

$$T_j = \binom{23}{j}\left(\frac{2}{5}\cdot\frac{3}{2}\right)^j = \binom{23}{j}\left(\frac{3}{5}\right)^j$$

$$\left|\frac{T_j}{T_{j-1}}\right| = \frac{24-j}{j}\times\frac{3}{5} \geq 1$$

$\Rightarrow j \leq 9$

Since ratio is exactly $1 \Rightarrow |T_8| = |T_9|$

$\Rightarrow k = 9 \rightarrow$ want max sum.

Q.175

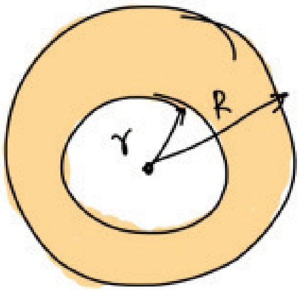
$z_1 = 4e^{i\theta}, z_2 = 12 + 6e^{i\phi}$

$$\omega = \underbrace{\frac{12k}{k+1}}_{z_0} + \underbrace{\frac{4}{k+1}}_{\text{Radius } r_1} e^{i\theta} + \underbrace{\frac{6k}{k+1}}_{\text{Radius } r_2} e^{i\phi}$$

$$\boxed{\omega = \frac{z_1 + kz_2}{1+k}}$$

Region S_k:

$$|r_1 - r_2| \le |\omega - z_0| \le r_1 + r_2$$



$$R = \frac{4+6k}{k+1}$$

$$r = \frac{|6k-4|}{k+1}$$

$$A(k) = \pi(R^2 - r^2) = \pi((r_1 + r_2)^2 - (r_1 - r_2)^2)$$

$$= 4\pi r_1 r_2$$

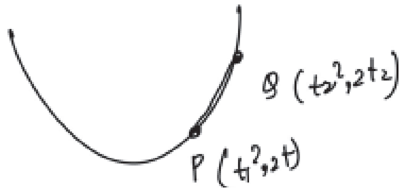
$$= \frac{96\pi k}{(k+1)^2}$$

$$= \frac{96\pi}{\underbrace{k + \frac{1}{k} + 2}_{\ge 1}}$$

since $k + \frac{1}{k} \ge 2$

$= 24\pi$ (max)

Q.176



$\overrightarrow{PQ} = (t_2^2 - t_1^2)\hat{i} + 2(t_2 - t_1)\hat{j}$

$\hat{n} = \dfrac{\hat{i} + \hat{j}}{\sqrt{2}}$

$|\overrightarrow{PQ} \cdot \hat{n}| = 10\sqrt{2}$

$\Rightarrow |t_2 - t_1||t_1 + t_2 + 2| = 20$

$\boxed{y = t_1 + t_2} \rightarrow \text{midpoint}$

$(t_2 - t_1)^2 = (t_1 + t_2)^2 - 4t_1t_2$

$\Rightarrow (t_2 - t_1)^2 = 4(x - \dfrac{y^2}{4}) = 4x - y^2$

So, $\boxed{(t_2 - t_1)^2(t_1 + t_2 + 2)^2 = 400}$

$\downarrow$ gives

$\boxed{(y^2 - 4x)(y + 2)^2 + 400 = 0}$

Q. 177

Let $g(x) = f(x^2)$

$\Rightarrow g(x) + g(y) = g\left(\dfrac{xy - 1}{x + y}\right)$ for $x, y \in \mathbb{R}$

$h(\theta) = \cot\theta$ (See $-$ feel!)

$h(\alpha) + h(\beta) = h(\alpha + \beta)$ for all $\alpha, \beta \in \mathbb{R}$

$\Rightarrow \boxed{h(\theta) = C\theta, C \in \mathbb{R}}$

$f(\cot^2\theta) = C\theta$, Solve $\cot^2\theta_0 = 0$

$\Rightarrow \theta_0 = \dfrac{\pi}{2} + n\pi, \theta_1 = \dfrac{\pi}{4} + n\dfrac{\pi}{2}.$

Since f is continuous, so, $\theta \in \left[\dfrac{k\pi}{2}, (k+1)\dfrac{\pi}{2}\right]$ for some integer k.

So we must have $|\theta_1 - \theta_0| \le \pi/2$ that forces n = 0, -1.

If n=0, $\theta_0 = \dfrac{\pi}{2}$ and $\theta_1 = \dfrac{\pi}{4}$ give $c = \dfrac{4038}{\pi}$.

Thus $\theta \in [0, \pi/2] \Rightarrow$ solving $\cot^2\theta = 3$ gives

$\theta = \frac{\pi}{6}$. Hence $f(3) = f(\cot^2 \frac{\pi}{6}) = \frac{4038}{\pi} \cdot \frac{\pi}{6} = 673$

Hence (if $n = -1, \theta_0 = \frac{-\pi}{2}, \theta_1 = \frac{-\pi}{4}, C = \frac{4038}{\pi}$ with $\theta \in [-\frac{\pi}{2}, 0]$, giving $f(3) = 673$ as well!)

Q.178 Equation of planes:

$P_1\{(2,4,0),(5,1,0),(3,2,10)\}$

$P_1 : \boxed{\frac{x}{6} + \frac{y}{6} + \frac{z}{60} = 1}$

$P_2\{(0,0,0),(2,4,0),(3,2,10)\}$

$P_2 : \boxed{x - \frac{y}{2} - \frac{z}{5} = 0}$

$P_3\{(0,0,0),(5,1,0),(3,2,10)\}$

$P_3 : \boxed{x - 5y + \frac{7}{10}z = 0}$

Using this equation $(\frac{12}{5}, \frac{23}{10}, 13)$ is on P_1, $(\frac{12}{5}, \frac{23}{10}, \frac{25}{4})$ is on P_2 and $(\frac{12}{5}, \frac{23}{10}, 13)$ on P_3. Only the lowest is on tetrahedron, so $\alpha = \frac{25}{4}$ and $\beta = \frac{5}{2}$. So,

$\max |\alpha - \beta| = \frac{15}{4}$

Q.179

For any such f, let $A=\{n|\ f(n)=n\}$ be the set of elements fixed by f and let $B=\{n|\ f(n)\in A\}$ be set of elements that are sent to A, but are not in A.

Let $C=\{1,2,3,4,5\}\setminus(A\cup B)$ be everything else.

Note that any possible value of f(f(x)) is in A, so A is non-empty. Cases of A is discussed.

If A has one-element, $\text{total}=5\left(\sum_{n=1}^{4}\binom{4}{n}n^{4-n}\right)$

If A has two element, $\text{total}=\binom{5}{2}\left(\sum_{n=1}^{3}\binom{3}{n}2^{n}n^{3-n}\right)$

If A has 3-element, $\text{total}=\binom{5}{3}\left(\sum_{n=1}^{2}\binom{2}{n}3^{n}n^{2-n}\right)$

If A has 5-element, $\text{total}=1$

Q.0180

$\boxed{\alpha^3=\alpha+1}$

Newton's idea gives sum = -15.

Q. 181

$$\boxed{\begin{aligned} n_1&=0\\ n_2&=10\\ n_3&=0\end{aligned}}$$

use Rolle's theorem.

$(x^2-1)^2=f$

$f'\text{has }3-\text{roots}:\ \left(\text{one in}(-1,1)\right)$

$f''\text{has }5-\text{roots}\ \left(2\text{ in}(-1,1)\right)$

Q.182

$$f'(x) = f'(x) + \int_0^x e^{x-y} f'(y)dy - (x^2 + x)e^x$$

$$\Rightarrow f(x) + (x^2 - x + 1)e^x - (x^2 + x)e^x = 0$$

$$\boxed{\Rightarrow \mathbf{f(x) = (2x-1)e^x}}$$

Q.183

$$\int f(1-g)dx = \int f dx - \int fg dx = 1/2 - \int fg dx$$

$$\int_0^1 fg dx \le \left(\int_0^1 f^2 dx\right)^{1/2} \left(\int_0^1 g^2 dx\right)^{1/2}$$

$$\int_0^1 (f(x) - f(1/2))^2 dx = \int_0^1 f(x)^2 dx - f(1/2) + f(1/2)^2$$

$$\Rightarrow \int_0^1 (f(x) - f(1/2))^2 dx \le \int_0^1 (x - 1/2)^2 dx = \frac{1}{12}$$

$$\Rightarrow \int_0^1 f^2 dx \le \frac{1}{12} + f(1/2) - f(1/2)^2$$

$$\le \frac{1}{12} + \frac{1}{2} - \frac{1}{4} \le \frac{1}{3}$$

Since, $y - y^2$ is maximized at $y = 1/2$.

$$\int_0^1 fg dx = 1/2 - \int_0^1 f(1-g)dx > 1/2 - 1/3 = 1/6$$

$$\therefore \int_0^1 fg dx \le \int_0^1 \frac{f^2 + g^2}{2} dx = \frac{1}{2}\int_0^1 f^2 dx + \frac{1}{2}\int_0^1 g^2 dx$$

$$(\text{A.M.} - \text{G.M.}) \le 1/6 + 1/6 \le 1/3$$

Q.184

$$P_n = (1-2^{1-n})^{2^n}$$

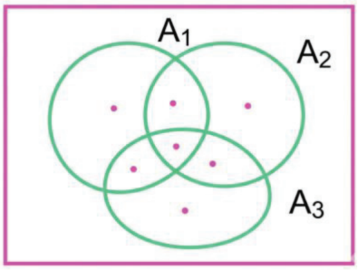
$$\lim_{n\to\infty}\left(1+\frac{-2}{2^n}\right)^{2^n}, \text{ let } 2^n = x$$

$$= \lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x$$

$$= \lim_{x\to\infty} e^{x\times\frac{-2}{x}} = e^{-2} = \frac{1}{e^2}$$

For n = 3

$S = \{1, 2, 3, 4, 5, 6, 7, 8\}$



We want:

$A_1 = \{1, 2, 3\}$

$A_2 = \{1, 3,4,5\}$

$A_3 = \{5,6,7,8\}$

1 has 6 choice and so is others!

$$P = \frac{(2^3-2)(2^3-2)\cdots(2^3-2)}{2^3\times 2^3\times\cdots\times 2^3} = \frac{(2^3-2)^8}{(2^3)^8}$$

$$= (1-2^{1-3})^{(2^3)}$$

In general:

$$P_n = (1-2^{1-n})^{2^n}$$

Q.185

$$\sum_{n=0}^{\infty}\frac{\cos(\frac{\pi n}{6})}{\sqrt{3}^{n}}=\frac{1}{3^{0}}+\frac{\sqrt{3}/2}{3^{1/2}}+\frac{1/2}{3^{1}}+\frac{0}{3^{3/2}}-\frac{1/2}{3^{2}}-\frac{\sqrt{3}/2}{3^{5/2}}$$

=3/2

(Sum is 6 different G.P. with different first term but same common ratio of $\frac{-1}{27}$)

Q.186

$$M=\begin{bmatrix}1 & 2 & 3\\ 2 & 2 & 1\\ 3 & 3 & 1\end{bmatrix}$$

$$|M|=1 \text{ and } M^{-1}=\begin{bmatrix}-1 & 7 & -4\\ 1 & -8 & 5\\ 0 & 2 & -2\end{bmatrix}$$

This matrix maps given T that has same number of lattice pts in or on boundary. The vertices map to (0,0,0), (10,0,0), (0,10,0) and (0,0,10).

$$N=\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{12}{2}$$

$$=\binom{13}{3}=286$$

(COOL PROBLEM!)

Q. 187

$$y=mx+\frac{a}{m}$$

$$\Rightarrow -y_1m+m^2x_1+a=0$$

$$\Rightarrow m_1+m_2=\tan\theta+\tan\phi=\frac{y_1}{x_1}=b$$

So P lies on $y=bx$ for any $a\in\mathbb{R}-\{0\}$

$y=2025x$ for $b=2025$

$$\Rightarrow \beta=4050$$

So,

$$\left|\frac{\beta-4030}{4}\right|=5$$

Q.188

$(r_1, r_2) = (1,2) \quad d = 5 \quad M = \left(\frac{3}{2}, 2\right)$

$$\begin{cases} R_1 = \dfrac{d^2}{4} + r_1 r_2 \\ R_2 = \dfrac{d^2}{4} - r_1 r_2 \end{cases} \rightarrow \text{Proof below!}$$

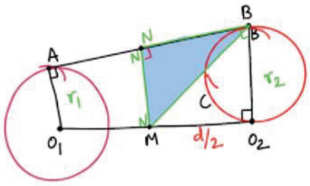
$L : S_1 - S_2 = 0 \Rightarrow 3x + 4y - 11 = 0$

(P) $4\left(\frac{d^2}{4} + r_1 r_2 - \left(\frac{d^2}{4} - r_1 r_2\right)\right) = 8 r_1 r_2 = 16$

(Q) $r_1 r_2 = 2\left((MO)^2 - R_2^2\right)$

(R) $d(O_1, L) = \frac{11}{5}$

(S) $d(M, L) = \frac{3}{10}$



Proof:

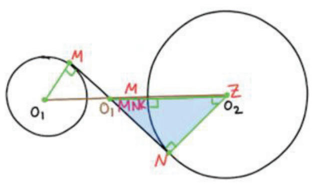
$MN \parallel O_1A \parallel O_2B$

$MN = \frac{r_1 + r_2}{2}$ (M is mid point)

$(MB)^2 = R_1^2 = \frac{d^2}{4} + r_1 r_2$ (using pythagoras theorem)

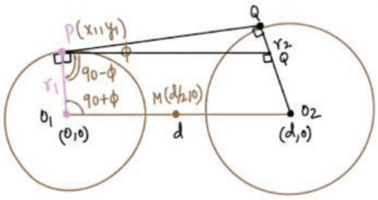
$PQ = \frac{|r_1 - r_2|}{2}$

ΔPGN (use pythagoras) $(MN)^2 = R_2^2 = \frac{d^2}{4} - r_1 r_2$



Final Result $\boxed{\begin{aligned} R_1^2 &= \frac{d^2}{4} + r_1 r_2 \\ R_2^2 &= \frac{d^2}{4} - r_1 r_2 \end{aligned}}$

(Note: You can also use Coordinate Geometry as follows!)



$\boxed{\begin{array}{l} x_1 = r_1\cos(90^\circ + \phi) \\ y_1 = r_1\sin(90^\circ + \phi) \end{array}}$ $\quad \sin\phi = \dfrac{r_2 - r_1}{d}$

$$(PM)^2 = \left(x_1 - \frac{d}{2}\right)^2 + y_1^2 = \frac{d^2}{4} + r_1 r_2 \left(\text{using: } x_1^2 + y_1^2 = r_1^2\right)$$

(Similarly do others)

Q.189

$\vec{u}, \vec{v}, \vec{\omega}$ are mutually orthogonal.

$(\vec{v}\times\vec{\omega})\times\vec{v} = \vec{\omega} \Rightarrow |\vec{v}|^2 = 1$

$(\text{as}(|\vec{v}|^2\,\vec{\omega} - (\vec{v}\cdot\vec{\omega})\vec{v} = \vec{\omega})$

$|\vec{\omega}| = 3$ and $|\vec{u}| = |\vec{v}||\vec{\omega}| = 3$ (as orthogonal).

$\boxed{\Delta = -(t+1)^2(t+2)}$

Since $t \neq 1$, we must have t=-2

Q.190

(A) $y^2 = 4x, \quad y = Mx - 2M - M^3 \quad \left(M = -\dfrac{1}{m}\right)$

Comparing $\boxed{m^2 = \dfrac{-l^3}{2l + 12}}$

4 points $((2,1),(2,-1),(-4,4),(-4,-4))$

(B) Normal must pass through centre $(0,1)$

(C) Normal condition: $\dfrac{a^2}{l^2} + \dfrac{b^2}{m^2} = \dfrac{(a^2 - b^2)^2}{n^2}$

(Derive by Comparing parametric form)

(D) $\boxed{ax\cos\theta + by\cot\theta = a^2 + b^2} \quad \rightarrow$ Normal

$\Rightarrow 3x\cos\theta + 2y\cot\theta = 13((\sqrt{13},0)$ passes)

$\Rightarrow \cos\theta = \dfrac{13}{3\sqrt{13}}$ (No sol^n)

[But one sol^n is there clearly the axis. This happened because at $\theta = 0, \cot\theta$ is undefined and inherently we are doing all for $\theta \neq 0$.

Observe: $(\sqrt{13},0)$ is Focus !]

Q.191

Since y is continuous and differentiable $\Rightarrow y(0)=0$

$$\lim_{x\to 0^+}\frac{y(x)}{x}=y'(0)$$

$$y'(0)=\lim_{x\to 0^+}\left(\alpha\frac{\sin(\beta x)}{x^{\gamma}}-2.0\right)$$

$$=\alpha\lim_{x\to 0^+}\left(\frac{\sin\beta x}{x^{\gamma}}\right)$$

P. $w=(1,\pi,1)\Rightarrow L=\lim_{x\to 0^+}\frac{\sin\pi x}{x}=\pi$

Q. $w=(2,\pi,1)\Rightarrow L=2\pi$

R. $w=(10,\pi,0)\Rightarrow L=\lim_{x\to 0^+}\sin(10\pi)=0$

S. $w=(1,\pi,2)\Rightarrow L=\lim_{x\to 0^+}\frac{\sin\pi x}{x^2}\to\infty$(D.N.E.)

(Note: Since we defined $L:T\to\mathbb{R}\cup\{\infty\}$, a limit $\to\infty$ is a valid option (calibrating L to be a function instead of "Does not exist").

∞ is NOT a real number, so we use $\mathbb{R}\cup\{\infty\}$ to include it.

(JEE-maths is well known for Notations! Hence problem discusses that!)

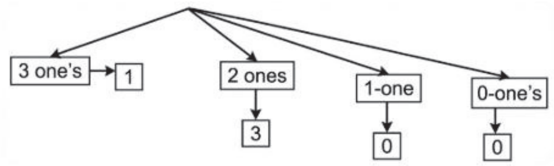
Q. 192

P. $|I - AB| = |I - BA|$ (Sylvester's)

R. $|I - X| = 0 \Rightarrow (1-x_1)(1-x_2)(1-x_3) = 0$ at least one must be 1 among x_i.

Total - No one's = $2^3 - 1 = 7$

Q. $x_1 + x_2 + x_3 = x_1x_2x_3 + 2$



Total =4

S. $tr(I - X) = 3 \Rightarrow tr(I) - tr(X) = 3$

$\Rightarrow tr(X) = 0 \rightarrow$ One matrix (NULL)

Sylvester's Beautiful Proof!

Statement: For A and B be n× n Matrix then if

(I - AB) { has inverse} $\Rightarrow$ (I - BA) has inverse TOO.

As

Then $\exists \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that $(I - BA)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

As $\boxed{\begin{array}{l} \sqcup x + \sqcup y + \sqcup z = 0 \\ \odot x + \# y + \equiv z = 0 \\ \uparrow x + \downarrow y + \otimes z = 0 \end{array}}$

Homogeneous system with zero det.

Hence, $0 = A\mathbf{0} = A(I - BA)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (A - ABA)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (I - AB)A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

But it means $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ as $|I - AB| \neq 0$ (inverse of (I - AB) exists in question)

Since it is true for all A here, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ which contradicts $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0$.

M-2: Let $X = (I - AB)^{-1} \Rightarrow \boxed{X(I - AB) = I}$

$I = I - BA + BA = I - BA + BX(I - AB)A$

$= I - BA + BXA(I - BA)$

$= (I + BXA)(I - BA)$

$\Rightarrow I + BXA$ is the inverse of I - BA.

$\boxed{(I - BA)^{-1} = I + B(I - AB)^{-1}A}$ $\rightarrow$Remember

Note: $|I - AB| = |I - BA|$

But $tr(AB) = tr(BA)$

While $tr(I - AB) \neq tr(I - BA)$ (cool!)

(Reference: PROBLEMS in THIN BOOK WITH FAT IDEAS)

Q.193

$Tr(A^n)$ we need to take the characteristic equation:

$$\begin{vmatrix} 2\cos\theta-\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-2\lambda\cos\theta+1)=0$$

$1, e^{i\theta}, e^{-i\theta}$ (roots eign values)

Since $AX=\lambda X$ (characteristic eqn)

$A^2X=\lambda^2X$

.

.

$\boxed{A^nX=\lambda^nX}$

Sum of roots will give Trace!

$A \to$ has $\lambda_1, \lambda_2, \lambda_3$ roots

$Tr(A^n)=\lambda_1^n+\lambda_2^n+\lambda_3^n$ (cool!)

Proof: $A=PDP^{-1}$ (P is by eigen vector and D is diagonal matrix)

$A^n=PD^nP^{-1}$

$Tr(A^n)=Tr(PD^nP^{-1})=Tr((P^{-1}P)D^n)$

$=Tr(D^n)$ (Done!)

Hence, $T_n=1+e^{in\theta}+e^{-in\theta}=1+2\cos(n\theta)$

$1+2\cos(n\theta)=1 \Rightarrow \boxed{\cos(n\theta)=0}$

For $\theta \in (0, 2\pi)$ if $\cos(k\theta)=0$

$$\Rightarrow \theta=\frac{(2m+1)\pi}{2k}, 0<\frac{(2m+1)\pi}{2k}<2\pi$$

$\Rightarrow \boxed{0<2m+1<4k}$

$\boxed{n(S_k)=2k}$

Now: $n(A\cup B)=n(A)+n(B)-n(A\cap B)$

Let $\theta \in S_a \cap S_b$. Then

$$\theta=\frac{(odd)_1\pi}{2a}=\frac{(odd)_2\pi}{2b}$$

$$\Rightarrow \boxed{\frac{(odd)_1}{a}=\frac{(odd)_2}{b}}$$

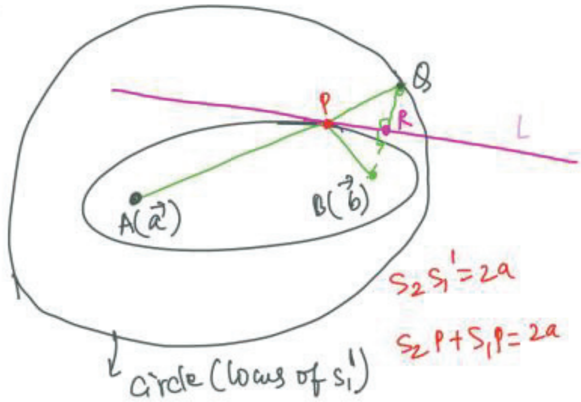
(A) $n(S_3\cup S_5)=6+10-2=14$

(B) $n(S_3\cup S_9)=6+18-6=18$

(C) $n(S_4\cup S_8)=8+16-0=24$

(D) $n(S_6\cup S_{10})=12+20-4=28$

Q.194

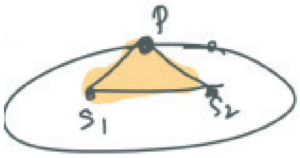


L is perpendicular bisector

1. Locus is Ellipse (T):

(A) $|\vec{b} - \vec{a}| = 10 = 2ae$ and $2a = 14 \Rightarrow a = 7$

$b^2 = a^2(1 - e^2) = 24$ and $e = \frac{5}{7}$

(C) 

$\max(\triangle APB) = \frac{1}{2} \times 2ae \times b = 5\sqrt{24}$

(D) Sum of Focal distance = 14

Linked Comprehension-1

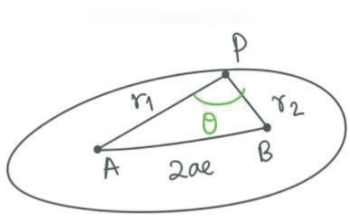
Q.195

(A) Perpendicular bisector is always tangent.

(B) $h_1h_2 = b^2 = 24$ (standard property of ellipse)

Think of power of point theorem! Feel!

(C)



Cosine Rule

$$(2ae)^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta$$

$$\Rightarrow 4a^2e^2 = (r_1 + r_2)^2 - 2r_1r_2(1+\cos\theta)$$

$$\Rightarrow 4a^2e^2 = 4a^2 - 4r_1r_2\cos^2\frac{\theta}{2}$$

$$\Rightarrow r_1r_2\cos^2\frac{\theta}{2} = a^2(1-e^2) = b^2 = 24$$ always TRUE.

(D) Locus of Foot of $\perp$ to tangent from Focus lies on auxiliary circle (Proof using mid-point theorem!)

Linked Comprehension-2

Q.196

$$2a_{n+1} - 5a_n = \sqrt{21a_n^2 + 4}$$

$$\Rightarrow \boxed{4a_{n+1}^2 - 20a_{n+1}a_n + 25a_n^2 = 21a_n^2 + 4}$$

$$\Rightarrow \boxed{4a_{n+1}^2 - 20a_{n+1}a_n + 4a_n^2 = 4}$$

$$a_n^2 - 5a_n a_{n-1} + a_{n-1}^2 = 1 \rightarrow \text{Put} \boxed{n \rightarrow n-1}$$

Subtracting both we get:

$$(a_{n+1} - a_{n-1})(a_{n+1} + a_{n-1} - 5a_n) = 0$$

$a_{n+1} \neq a_{n-1}$ sequence is increasing

$\boxed{\Rightarrow a_{n+1} = 5a_n - a_{n-1}}$ since $a_1 = 5, a_2 = 24, a_0 = 1$

Since,

$$\boxed{a_n^2 - 5a_n a_{n-1} + a_{n-1}^2 = 1}$$

Quadraticin a_n with roots $\rightarrow a_{n+1}, a_{n-1}$

$$a_{n+1}^2 - (5a_n)a_{n+1} + a_n^2 = 1$$

$$\Rightarrow a_{n+1}^2 - (a_{n+1} + a_{n-1})a_{n+1} + a_n^2 = 1$$

$$\Rightarrow \boxed{a_n^2 - a_{n+1}a_{n-1} = 1}$$

$$\sum \frac{1}{a_n a_{n+1}} = \sum \left(\frac{a_n^2 - a_{n+1}a_{n-1}}{a_n a_{n+1}} \right)$$

$$= \sum \left(\frac{a_n}{a_{n+1}} - \frac{a_{n-1}}{a_n} \right) \rightarrow \text{Telescoping}$$

To find $\alpha = \dfrac{a_n}{a_{n+1}}$ at $n \rightarrow \infty$, let the ration be α

$$x^2 - 5x + 1 = 0 \Rightarrow \boxed{\alpha = \frac{5 - \sqrt{21}}{2}}$$

Q.197

$$\boxed{\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=\frac{a_{n-1}}{a_n}=\frac{a_{n+1}}{a_{n+2}}=\ldots}$$

$$S=\alpha-\frac{1}{5}=\frac{23-5\sqrt{21}}{10}$$

Linked Comprehension-3

Q.198

$$(1+x)^n=a_n+b_nx+c_nx^2+\ldots$$

$$x=1,\quad (1+1)^n=a_n+b_n+c_n$$

$$x=\omega,\quad (1+\omega)^n=a_n+b_n\omega+c_n\omega^2$$

$$x=\omega^2,\quad (1+\omega^2)^n=a_n+b_n\omega^2+c_n\omega$$

(1) $E=a_n^3+b_n^3+c_n^3-3a_nb_nc_n$

$$=(a_n+b_n+c_n)(a_n+b_n\omega+c_n\omega^2)(a_n+b_n\omega^2+c_n\omega)$$

$$=2^n((1+\omega)(1+\omega^2))^n$$

$$=2^n \quad \boxed{\text{at}\, n=26, E=2^{26}}$$

Q.199

$$a_n^2 + b_n^2 + c_n^2 - a_n b_n - b_n c_n - c_n a_n$$
$$= (a_n + b_n\omega + c_n\omega^2)(a_n + b_n\omega^2 + c_n\omega)$$
$$= |a_n + b_n\omega + c_n\omega^2|^2$$
$$= |(1+\omega)^n|^2 = 1$$

So, Sum(S) = 2 · 1 = 2

Extra:

$$a_{26} + b_{26}\omega + c_{26}\omega^2 = (1+\omega)^{26} = \omega$$

$$\Rightarrow (a_{26} - c_{26}) + (b_{26} - c_{26})\omega = 0 + \omega$$

$$\Rightarrow \boxed{a_{26} = c_{26}} \text{ and } \boxed{b_{26} = c_{26} + 1}$$

(Comparing real and Imaginary part)

Linked Comprehension-4

Q.200

$$\boxed{z^5 + z + 1 = 0}$$

Factor of $1 \to 1, \omega, \omega^2$ OR $e^{i\frac{2\pi}{n}\cdot k}, n \in N$

$(z^2 + z + 1)(z^3 - z^2 + 1) = z^5 + z + 1$ (Factor theorem general!)

$$S = \{\omega, \omega^2, \alpha, \bar{\alpha}, \gamma\}, \gamma \in \mathbb{R}$$

$$\sum_{z\in S}(z^2 - z) = \boxed{(\omega^2 - \omega) + (\omega^4 - \omega^2)} + (\alpha^2 - \alpha) + (\bar{\alpha}^2 - \bar{\alpha}) + (\beta^2 - \beta)$$

$$= (\alpha^2 - \alpha) + (\bar{\alpha}^2 - \bar{\alpha}) + (\beta^2 - \beta), \beta \in \mathbb{R}$$

Let $f(x) = x^3 - x + 1$ has a real root $\gamma \in (-1, 0)$

$$\sum(\alpha^2 - \alpha) = (\sum\alpha)^2 - 2(\sum\alpha\beta) - \sum\alpha$$ Total sum =0
$$= 1 - 2\times 0 - 1 = 0$$

Q. 201

$\arg(\omega)+\arg(\omega^2)=0$

$\text{Realroot } \gamma<0 \Rightarrow \arg(\gamma)=\pi$

$\arg(\bar{\alpha})+\arg(\alpha)=0$ (as conjugates)

Q202.

Answer Key: (A), (B), (C)

The volume of the parallelepiped generated by the vectors $\vec{a}, \vec{b}$, and $\vec{c}$ is given by the absolute value of their scalar triple product, $V(\vec{a},\vec{b},\vec{c})=|(\vec{a}\times\vec{b})\cdot\vec{c}|$. Evaluating the cross product of the base vectors yields $\vec{a}\times\vec{b}=(\hat{i}+\hat{j}+2\hat{k})\times(-\hat{i}+3\hat{j}+\hat{k})=-5\hat{i}-3\hat{j}+4\hat{k}$. The scalar triple product evaluates to $[\vec{a}\ \vec{b}\ \vec{c}]=(-5)(1)+(-3)(2)+(4)(3)=1$, making the initial volume $V(\vec{a},\vec{b},\vec{c})=1$. The volumetric constraint for the locus $\mathcal{L}$ dictates that $V(\vec{a},\vec{b},\vec{r})=\frac{9}{10}(1)$, which algebraically translates to $|(\vec{a}\times\vec{b})\cdot\vec{r}|=0.9$. Substituting the Cartesian representation of $\vec{r}$, we obtain the equation $|-5x-3y+4z|=\frac{9}{10}$. Squaring both sides to eliminate the absolute value yields the Cartesian representation $(5x+3y-4z)^2=\frac{81}{100}$, which simplifies directly to the surface equation $(50x+30y-40z)^2=81$. This confirms statement (B). The absolute value equation $|5x + 3y - 4z| = 0.9$ represents a pair of planes, $5x + 3y - 4z = 0.9$ and $5x + 3y - 4z = -0.9$, which are distinctly parallel to the plane containing the origin and vectors $\vec{a}$ and $\vec{b}$ (given by $5x + 3y - 4z = 0$) and symmetrically distributed across it, validating statement (A). The orthogonal distance from the plane containing the base vectors to any point on $\mathcal{L}$ is found by evaluating the perpendicular distance from the origin to either plane in the locus, calculated as $d=\frac{0.9}{\sqrt{5^2+3^2+(-4)^2}}=\frac{0.9}{\sqrt{50}}=\frac{9}{50\sqrt{2}}$, rendering statement (C) correct.

The magnitude of the orthogonal projection of the vector $\vec{r}$ along the normal vector $(\vec{a}\times\vec{b})$ is simply the geometric altitude of the newly formed parallelepiped, computed by $\frac{|(\vec{a}\times\vec{b})\cdot\vec{r}|}{|\vec{a}\times\vec{b}|}$. We already established the numerator is 0.9 and the denominator is $\sqrt{50}$, making the projection magnitude exactly $\frac{9}{50\sqrt{2}}$. This indicates that statement (D) provides an incorrect value.

Q203. **Answer Key: (B)**

To find the maximum Euclidean distance between the regions $\mathcal{R}_1$ and $\mathcal{R}_2$, we first analyze their respective geometric boundaries.

The region $\mathcal{R}_1$ is bounded by the upward-opening semi-parabola $y = \sqrt{2x}$ and the upper semicircle of $x^2 + y^2 = 4x$. By completing the square, the circular boundary is $(x-2)^2 + y^2 = 4$, which is a circle centered at C(2,0) with a radius of R = 2. The condition $x^2 + y^2 \le 4x$ ensures that every point $P \in \mathcal{R}_1$ lies inside or on this circle. Therefore, the distance from any point $P \in \mathcal{R}_1$ to the center C(2,0) satisfies the strict upper bound $PC \le 2$.

The region $\mathcal{R}_2$ is bounded by the x-axis $(y \le 0)$, the line y = 1 - x, and the parabola y = (x - 1)(x - 4). The vertices of this region are located at the intersections of these boundaries: $V_1(1,0), V_2(4,0)$, and the intersection of the line and the parabola given by $1 - x = x^2 - 5x + 4 \Rightarrow x^2 - 4x + 3 = 0$, yielding x = 1, 3. The corresponding vertex in the lower half-plane is $V_3(3, -2)$.

We aim to maximize the distance PQ. By the triangle inequality, for any $P \in \mathcal{R}_1$ and $Q \in \mathcal{R}_2$, the distance is bounded by the sum of their distances to the center C(2,0):

$PQ \le PC + CQ$

Since $PC \le 2$ for all $P \in \mathcal{R}_1$, we have $PQ \le 2 + CQ$. To maximize this bound, we must find the maximum value of CQ for $Q \in \mathcal{R}_2$. Evaluating the distance squared from C(2,0) to the vertices of $\mathcal{R}_2$:

$$CQ_1^2 = (2-1)^2 + (0-0)^2 = 1$$
$$CQ_2^2 = (2-4)^2 + (0-0)^2 = 4$$
$$CQ_3^2 = (2-3)^2 + (0-(-2))^2 = 1 + 4 = 5$$

Calculus verification confirms that the maximum distance from C(2,0) to any point in the compact convex-like region $\mathcal{R}_2$ occurs exactly at the vertex $V_3(3,-2)$

making the maximum value of $CQ = \sqrt{5}$.

Substituting this back into our inequality gives the theoretical upper bound

$PQ \le 2 + \sqrt{5}$. This maximum distance is strictly attainable if and only if there exists a point $P \in \mathcal{R}_1$ such that P, C, and V_3 , are collinear, with C situated between P and V_3, and P lying exactly on the circular boundary (PC = 2). The unit vector from $V_3(3,-2)$ to $C(2,0)$ is $\hat{u} = \frac{1}{\sqrt{5}}\langle -1,2\rangle$. The coordinates of the theoretical optimal point P are:

$$P = C + 2\hat{u} = \left(2 - \frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$$

To confirm this theoretical maximum is valid, we must verify that P satisfies all constraints of $\mathcal{R}_1$. Since $2 - \frac{2}{\sqrt{5}} > 0$ and $\frac{4}{\sqrt{5}} > 0$, P is strictly in the first quadrant. It lies on the circle by construction. We finally check the parabolic constraint $y^2 \geq 2x$:

$$y^2 = \left(\frac{4}{\sqrt{5}}\right)^2 = \frac{16}{5} = 3.2$$

$$2x = 2\left(2 - \frac{2}{\sqrt{5}}\right) = 4 - \frac{4}{\sqrt{5}} \approx 4 - 1.789 = 2.211$$

Since $3.2 \geq 2.211$ is strictly true, the point P lies perfectly within $\mathcal{R}_1$. Therefore, the absolute maximum distance between the two regions is exactly $2 + \sqrt{5}$.

Q204.

Core Concept: In 3D space, if any two lines in a set intersect, the entire configuration is restricted to two possible geometric states:

Planar Configuration: All lines lie within a single plane.

Concurrent Configuration: All lines pass through a single common point.

Step-by-Step Analysis:

Statement (A) & (C): The condition $[\vec{d}_i \vec{d}_j \vec{d}_k] \neq 0$ implies the direction vectors are non-coplanar. Since the lines are not coplanar but must intersect pairwise, they are forced to be concurrent. Both are **Correct**.

Statement (B): If the lines are not concurrent, they must be coplanar. In any plane, the scalar triple product of any three direction vectors is identically zero. **Correct**.

Statement (D): For n = 3, if they are not concurrent but intersect pairwise, they form a triangle or a set of lines in a plane. Thus, they must be coplanar. **Correct**.

Correct Options: (A), (B), (C), (D)

Q205.

Analysis of Options:

(A) Correct: As derived previously, $\text{Perimeter}(\triangle PF_1R) = 4a$ and $\text{Perimeter}(\triangle PF_2Q) = 4a$. Thus, their sum is 8a.

(B) Incorrect: The areas depend on the sine of the angle between the focal chords and the major axis. $\text{Area}(\triangle PF_1R) = \frac{1}{2}(PF_1)(PR)\sin\theta$. These areas are equal only if P is on the minor axis (symmetry) or if $F_1F_2 \parallel QR$.

(C) Correct: For any focal chord of an ellipse, the harmonic mean of the segments is the semi-latus rectum: $\frac{1}{PF_1} + \frac{1}{QF_1} = \frac{2a}{b^2}$. Applying this to both chords PQ and PR, the sum becomes $\frac{2a}{b^2} + \frac{2a}{b^2} = \frac{4a}{b^2}$.

(D) Correct: The line QR is parallel to the tangent at P only in specific symmetric cases. For the circle on diameter F_1F_2 (the director circle of a degenerate conic, but here centered at (2,3) with radius ae) to be tangent to QR, the distance from (2,3) to QR must be ae. This occurs when P is at the extremity of the minor axis.

Correct Options: (A), (C), (D)

Q206.

Rigorous Seamless Solution

Applying the change of variable $t = u^2$ to the integral defining $\Phi(x)$ yields the simplified form $\Phi(x) = \frac{1}{x}\int_0^x 2uf(u)du$. By the definition of the function, evaluating this identity at x = c for any c $\in (0,1)$ confirms $\Phi(c) = \frac{1}{c}\int_0^{c^2} f(\sqrt{t})dt$, and consequently, cross-multiplying gives $\int_0^c 2tf(t)dt = c\cdot\Phi(c)$, validating both **(B)** and **(D)** as fundamental consequences of the function's definition.

Differentiating the product $x^{-1}\cdot\int_0^x 2uf(u)du$ via the Leibniz Integral Rule and the standard product rule results in $\Phi'(x) = -\frac{1}{x^2}\int_0^x 2uf(u)du + \frac{1}{x}[2xf(x)]$. Substituting $\frac{\Phi(x)}{x}$ for the first term, we obtain the identity

$\Phi'(x) = 2f(x) - \frac{\Phi(x)}{x}$ for the entire domain (0, 1], which rigorously guarantees the existence of a point c satisfying $\Phi'(c) = 2f(c) - \frac{\Phi(c)}{c}$, proving **(A)**.

Finally, the limit in **(C)** is evaluated as a Riemann sum:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2} = \int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4}.$$

Since f is continuous and strictly increases from 0 to 1, the Intermediate Value Theorem ensures that f(x) takes every value in (0, 1); as $0 < \frac{\pi}{4} < 1$, there must exist a c $\in (0,1)$ such that f(c) $= \frac{\pi}{4}$, confirming **(C)**.

Final Answer: (A), (B), (C), (D)

Q207. **Rigorous Seamless Solution**

Substituting $t = u^2 \Rightarrow dt = 2u\,du$ into the definition of $\Phi(x)$ transforms the integral limits to [0, x], yielding $\Phi(x) = \int_0^x 2uf(u)\cos\left(\frac{\pi u}{2x}\right) du$. Replacing the dummy variable u with t and evaluating at x = c immediately validates **(D)** as a direct restatement of this identity. Differentiating $\Phi(x)$ with respect to x using the Leibniz Rule results in a boundary term $2xf(x)\cos(\pi/2)$ and an integral term; since $\cos(\pi/2) = 0$, the boundary term vanishes, leaving

$$\Phi'(x) = \int_0^x \frac{\pi u^2 f(u)}{2x^2}\sin\left(\frac{\pi u}{2x}\right) du.$$

Statement **(A)** is incorrect because $\Phi'(x)$ is a weighted average of f(u) values that, for a strictly increasing function, remains strictly less than 2xf(x).

Statement **(B)** equates $\Phi'(c)$ to $\Phi(c)/c$, which requires the derivative of $\Phi(x)/x$ to vanish; however, for the given strictly increasing f, $\Phi(x)/x$ is generally monotonic, making the existence of such a c non-guaranteed. Finally, calculating the integral in **(C)** gives $\int_0^1 \sin(\frac{\pi u}{2})\,du = \frac{2}{\pi}$. Since f is continuous and strictly increases from 0 to 1, the Intermediate Value Theorem ensures the existence of c $\in$ (0, 1) such that $f(c) = \frac{2}{\pi}$, confirming **(C)**.

Final Answer: (C), (D)

Q208.

The function $f(x) = x^2 \left|\cos\left(\frac{\pi}{x}\right)\right|$ is a composition of a smooth parabolic envelope and a rapidly oscillating modulus term. To evaluate its behavior at the origin, we observe that since $0 \le \left|\cos\left(\frac{\pi}{x}\right)\right| \le 1$, the function is bounded such that $0 \le |f(x)| \le x^2$. By the **Squeeze Theorem**, as $x \to 0$, the limit $\lim_{x\to 0} f(x)$ is forced to 0, which matches f(0), thereby confirming that **f(x) is continuous at x = 0**.

To investigate differentiability at the origin, we apply the first principle:

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2\left|\cos\left(\frac{\pi}{h}\right)\right|}{h} = \lim_{h\to 0} h\left|\cos\left(\frac{\pi}{h}\right)\right|$$

Given that $\left|\cos\left(\frac{\pi}{h}\right)\right|$ remains bounded, the linear factor h ensures the limit vanishes, meaning **f'(0) exists and equals 0**. This indicates that the graph is tangent to the x-axis at x = 0.

However, the nature of the function in the neighborhood of the origin is highly pathological. The modulus operator introduces non-differentiable "corners" or cusps whenever $\cos(\pi / x) = 0$, which occurs at $x_n = \frac{2}{2n+1}$ for $n \in \mathbb{Z}$. Since the sequence x_n has a limit point at 0, any neighborhood $(-\delta, \delta)$ contains an infinite number of these points where the left-hand and right-hand derivatives disagree in sign. Consequently, **f(x) is non-differentiable at infinitely many points near x = 0**.

Finally, for $x \ne x_n$, the derivative f'(x) involves a term proportional to $\pi\sin(\pi / x)$ arising from the chain rule. As x approaches 0, this term oscillates rapidly between $-\pi$ and π, preventing $\lim_{x\to 0} f'(x)$ from existing. Because the limit of the derivative does not exist, **f'(x) is discontinuous at x = 0**, despite the derivative existing at that point.

Final Answer: (A), (B), and (C)

Q209.

Rigorous JEE-Style Analysis

Analysis of f(x) (The Modified Dirichlet Function):

The function f(x) is defined by the density of rational and irrational numbers. For any a $\neq$ 0, every neighborhood $(a-\delta, a+\delta)$ contains both x $\in \mathbb{Q}$

(where f(x) $\to a^2$) and x $\notin \mathbb{Q}$ (where f(x) \to 0). Since $a^2 \neq$ 0, the limit

$\lim\limits_{x\to a}$ f(x) does not exist, rendering f discontinuous everywhere except possibly at the origin. At x = 0, we observe $|f(x)-0| \leq x^2$. For any $\epsilon > 0$, choosing

$\delta = \sqrt{\epsilon}$ ensures |f(x) - f(0)| < ϵ whenever |x| < δ. Thus, **f(x) is continuous only at x = 0.**

To test differentiability at x = 0, we examine the limit of the difference quotient:

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)}{h}$$

Applying the sandwich theorem to the inequality $0 \leq \left|\frac{f(h)}{h}\right| \leq \frac{h^2}{|h|} = |h|$, we find that the limit exists and equals 0. Consequently, f'(0) = 0 **exists**, making f(x) a classic example of a function differentiable at exactly one point.

Analysis of g(x) (The Weierstrass Function):

The series $g(x) = \sum_{n=0}^{\infty} u_n(x)$, where $u_n(x) = \frac{\cos(3^n x)}{2^n}$, is governed by the **Weierstrass M-Test**. Since $|u_n(x)| \leq \frac{1}{2^n}$ and the geometric series $\sum \frac{1}{2^n}$ converges, the series for g(x) converges uniformly on $\mathbb{R}$. As each $u_n(x)$ is a continuous function, their uniform limit g(x) must also be **continuous for all** $x \in \mathbb{R}$.

However, the differentiability of g(x) fails everywhere. In this specific Weierstrass construction g(x) $= \sum a^n \cos(b^n x)$, the condition ab > 1 (here $3 \times \frac{1}{2} = 1.5 > 1$) ensures that the oscillations grow faster than the smoothing effect of the coefficients. As $n \to \infty$, the "spikes" in the function become infinitely sharp at every point x. This prevents the existence of a unique tangent line at any coordinate, classifying g(x) as a **nowhere differentiable function**.

Final Verdict:

Statements (A), (B), and (D) are Correct. Statement (C) is a partial truth but is technically subsumed and clarified by the "nowhere differentiable" property in (D). In a JEE Advanced multi-correct context, (C) is often included as a distractor or a subset of (D).

Correct Options: (A), (B), (C), and (D)

Q210.

Step 1: Domain Splitting

Consider the integral $J = \int_0^\infty \frac{x^{a-1}}{1+x} dx$. Splitting the domain at x=1:

$J = \int_0^1 \frac{x^{a-1}}{1+x} dx + \int_1^\infty \frac{x^{a-1}}{1+x} dx$ In the second integral, let $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2} dt.$
Changing the limits:

$$\int_1^\infty \frac{x^{a-1}}{1+x} dx = \int_1^0 \frac{t^{-(a-1)}}{1+\frac{1}{t}} \left(-\frac{1}{t^2}\right) dt = \int_0^1 \frac{t^{-a}}{1+t} dt$$

Thus, $J = \int_0^1 \frac{x^{a-1} + x^{-a}}{1+x}$ dx = g(a). **Option (A) is correct.**

Step 2: Symmetry

Replacing a with 1-a in g(a):

$$g(1-a) = \int_0^1 \frac{x^{(1-a)-1} + x^{-(1-a)}}{1+x} dx = \int_0^1 \frac{x^{-a} + x^{a-1}}{1+x} dx = g(a)$$

Since the function is symmetric about a = 1/2 and differentiable, g'(1/2) = 0.

Options (B) and (C) are correct.

Step 3: Specific Value at a = 1/2

$$g(1/2) = \int_0^\infty \frac{x^{-1/2}}{1+x} dx$$

Substitute $x = \tan^2\theta \Rightarrow dx = 2\tan\theta\sec^2\theta \, d\theta$:

$$g(1/2) = \int_0^{\pi/2} \frac{\cot\theta}{\sec^2\theta} (2\tan\theta\sec^2\theta) \, d\theta = \int_0^{\pi/2} 2 \, d\theta = \pi$$

Option (D) is correct.

Correct Options: (A, B, C, D)

Q211.

Using the identity $\sin(x-y)\sin(x+y)=\sin^2 x-\sin^2 y$, the general term T_k simplifies to the telescoping form:

$$T_k=\frac{\sin^2\theta_k-\sin^2\theta_{k-1}}{\cos^2\theta_k\cos^2\theta_{k-1}}=\tan^2\theta_k-\tan^2\theta_{k-1}$$

The sum f(n) telescopes to:

$$f(n)=\tan^2\left(\frac{(n-1)\pi}{2n}\right)-\tan^2(0)=\cot^2\left(\frac{\pi}{2n}\right)$$

Evaluating the limit: $\lim\limits_{n\to\infty}\frac{1}{n^2}\cot^2\left(\frac{\pi}{2n}\right)=\frac{4}{\pi^2}\Rightarrow a=4$

For $y^2 = 4x$, the latus rectum 4a' (where a' = 1) is 4. Thus, (B) is elegantly satisfied.

Correct Options: (A), (B), (C), (D)

Q212.

Let $B=\text{adj}(A)$. Given n = 3, $\det(B)=(\det A)^2$. Computing det(B):

det(B) = 2(18 - 12) + 2(-12 + 8) = 12 - 8 = 4 $\Rightarrow$ det A = $\pm$ 2

This confirms $\{\det(A):A\in S\}=\{-2,2\}$. Using the relation

$A=\frac{1}{\det A}\text{adj}(B)$, we find two distinct matrices A_1 and A_2 such that

$A_2 = -A_1$. Therefore, |S| = 2 and $\text{Tr}(A_1)+\text{Tr}(A_2)=0$.

For n = 3, the identity $\text{adj}(\text{adj}A)=(\det A)^{n-2}A$ reduces to $\text{adj}(\text{adj}A)=(\det A)A$.

Correct Options: (A), (B), (C), (D)

Q213.

1. Finding M (Maximum Roots):

Let $f(x) = e^x - P(x)$. Applying Rolle's Theorem repeatedly:

- $f^{(iv)}(x) = e^x$.Since $e^x \neq 0, f^{(iv)}(x)$ has 0 roots.
- By induction, the number of roots of $f^{(k)}(x)$ is at most one more than the number of roots of $f^{(k+1)}(x)$.
- Thus, f(x) can have at most 0 + 4 = 4 roots. Hence, **M = 4**.

2. Finding m (Minimum Roots):

Consider the end behavior of $f(x) = e^x - (x^3 + ax^2 + bx + c)$:

- $\lim_{x\to\infty} f(x) = \infty$ (Exponential dominance).

$\lim_{x\to-\infty} f(x) = 0 - (-\infty) = \infty.$

Since f(x) is continuous and $\lim_{x\to\pm\infty} f(x) = \infty$, the function can remain entirely positive (e.g., for $P(x) = x^3 - 10^6$). Thus, **m = 0**.

Correct Option: (B)

Q214.

- **Cauchy-Schwarz Equality:** The constraint $(\vec{a}\cdot\vec{c}) + (\vec{b}\cdot\vec{d}) = 6$ satisfies the equality condition $(6)^2 = (4)(9)$, implying $\vec{c} = \lambda\vec{a}$ and $\vec{d} = \lambda\vec{b}$. Comparing magnitudes gives \lambda = 1.5. (**A is Correct**)
- **Optimization:** Substituting $\lambda,$ we get $f = 3.25|\vec{a}\times\vec{b}|$. By AM-GM, $|\vec{a}||\vec{b}|$ is maximum when $|\vec{a}| = |\vec{b}| = \sqrt{2}$. For the cross product, we further require $\sin\theta = 1$. (**B is Correct**)
- **Range:** $f_{max} = 3.25 \times (\sqrt{2}\cdot\sqrt{2}\cdot\sin 90^\circ) = 3.25 \times 2 = 6.5$. (**C is Correct**)
- **Geometry:** Since $\vec{c} \,||\, \vec{a}$ and $\vec{d} \,||\, \vec{b}$, both vector pairs lie in the same plane. Their cross products are both normal to this plane and thus parallel. (**D is Correct**)

Correct Options: (A), (B), (C), (D)

Q215.

In polar coordinates with the focus as the pole, the focal distance is $r_k = \dfrac{2a}{1-\cos\theta_k}$. The symmetry of a regular n-gon centered at the origin implies that the sum of the projections onto any line is zero, thus $\sum_{k=1}^{n}\cos\theta_k = 0$, validating **(D)**.

Summing the reciprocals, we get $\sum \dfrac{1}{r_k} = \dfrac{1}{2a}\sum(1-\cos\theta_k) = \dfrac{1}{2a}(n-0) = \dfrac{n}{2a}$, validating **(A)**.

For any focal chord B_kC_k, the semi-latus rectum 2a is the harmonic mean of the segments FB_k and FC_k. Thus, $\dfrac{1}{FB_k} + \dfrac{1}{FC_k} = \dfrac{2}{2a} = \dfrac{1}{a}$, validating **(C)**.

Applying the **AM–HM Inequality** to the set $\{r_1, r_2, \ldots, r_n\}$, we have

$\dfrac{1}{n}\sum r_k \geq \dfrac{n}{\sum(1/r_k)} = 2a$. Since r_k is a function of $\cos\theta_k$ and θ_k varies for n ≥ 3, the distances are not all equal, making the inequality strict, $\overline{r} > 2a$, validating **(B)**.

Correct Options: (A), (B), (C), (D)

Q216.

Solution and Rigorous Mapping

- **(A) → (P):** $\Delta \neq 0$ ensures a non-degenerate conic. D = 0 implies the quadratic part is a perfect square, defining a **Parabola**.
- **(B) → (R):** $\Delta = 0$ represents a degenerate conic. D > 0 implies intersecting imaginary lines; the only real solution is the **Point** of intersection.
- **(C) → (S):** D > 0 indicates an elliptical type. The condition $J \cdot \Delta > 0$ ensures that the quadratic terms and the constant have the same sign, resulting in an **Empty Set**.
- **(D) → (T):** $\Delta = 0$ and D < 0 represent the degenerate case of two **Real Intersecting Lines**.
- **(E) → (Q):** $\Delta \neq 0$ and D < 0 represent a proper, non-degenerate **Hyperbola**.

Final Key: $(A)\to P, (B)\to R, (C)\to S, (D)\to T, (E)\to Q.$

Q217.

Rigorous Solution

- **(I) (P):** $\Delta \neq 0$ implies the conic is non-degenerate. D = 0 implies the quadratic part is a perfect square. This is the necessary and sufficient condition for a **Real Parabola**.
- **(II) $\rightarrow$ (Q):** $\Delta = 0$ is a degenerate conic. D > 0 implies the lines are imaginary but intersecting at a real point. The locus is a **Single Point**.
- **(III) $\rightarrow$ (R):** D > 0 indicates an elliptical type. If $J \cdot \Delta > 0$, the quadratic form and the shifted constant have the same sign, yielding no real solutions. The locus is an **Empty Set**.
- **(IV) $\rightarrow$ (R):** $\Delta = 0$ and D = 0 imply parallel lines. The sub-condition $g^2 + f^2 < Jc$ makes the distance between the lines imaginary. The locus is an **Empty Set** (not real parallel lines).
- Mapping (I) $\rightarrow$ (P) is True.
- Mapping (II) $\rightarrow$ (Q) is True.
- Mapping (III) $\rightarrow$ (R) is True.
- Mapping (IV) $\rightarrow$ (S) is **False** (it should be R).

Correct Options: (A), (B), and (D).

Q218.

In the interval $x \in [1, 2]$, the fractional part function is $\phi_2(x) = x - 1$. For the trigonometric function, since the angle πx lies in $[\pi, 2\pi]$, we have $\cos^{-1}(\cos \pi x) = 2\pi - \pi x$. Thus, $\phi_1(x) = \frac{1}{\pi}(2\pi - \pi x) = 2 - x$. These functions intersect at x = 1.5 where x - 1 = 2 - x.

The area of region Ω_1 is given by:

$$\text{Area}(\Omega_1) = \int_1^{1.5}(x-1)dx + \int_{1.5}^{2}(2-x)dx = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

Since g(x) is continuous on (1, 2) and its integral over the unit interval is 1/4, the function must attain the value 1/4 at some point $c \in (1, 2)$. This confirms **(A)**.

Region Ω_2 is the area bounded between f(x) and g(x). Its area is $\int_1^2 |f(x) - g(x)|dx = \int_1^2 |(2-x)-(x-1)|dx = \int_1^2 |3-2x|dx$.

Since |3 - 2x| = |2x - 3|, statement **(B)** is correct.

The union $\Omega_1 \cup \Omega_2$ represents the total area bounded by y = f(x) and the x-axis from x = 1 to x = 2. Therefore, $\text{Area}(\Omega_1 \cup \Omega_2) = \int_1^2 f(x)dx$, which validates **(C)**.

Noting that f(x) + g(x) = (2 - x) + (x - 1) = 1 for all $x \in S$, the total area under f(x) is 1 - $\text{Area}(\Omega_1)$ = 1 - 1/4 = 3/4. The ratio is:

$$\frac{3/4}{1/4} = 3$$

This confirms statement **(D)**.

Correct Options: (A), (B), (C), (D)

Q219.

1. Deriving the Functional Form

For $t \in (0, \pi)$, we utilize the identity 1 + cos t = 2 $\cos^2(t/2)$. Since cos(t/2) > 0 on this interval:

$$\sqrt{1+\cos t} = \sqrt{2}\cos(t/2)$$

Substituting this into the definition of f(x):

$$f(x) = 3\sqrt{2}\int_0^x \frac{\sqrt{2}\cos(t/2)}{17-8(2\cos^2(t/2)-1)}dt = 6\int_0^x \frac{\cos(t/2)}{25-16\cos^2(t/2)}dt$$

Using $\cos^2(t/2) = 1 - \sin^2(t/2)$, we get:

$$f(x) = 6\int_0^x \frac{\cos(t/2)}{9+16\sin^2(t/2)}dt$$

Let $u = \sin(t/2) \Rightarrow du = \frac{1}{2}\cos(t/2)dt.$

The limits transform to $u \in [0, \sin(x/2)]$:

$$f(x) = 12\int_0^{\sin(x/2)} \frac{du}{3^2+(4u)^2} = 12\left[\frac{1}{3\cdot 4}\arctan\left(\frac{4u}{3}\right)\right]_0^{\sin(x/2)}$$

$$f(x) = \arctan\left(\frac{4}{3}\sin\frac{x}{2}\right)$$

Hence, **Statement (C) is TRUE**.

2. Finding x_0

Given $\tan(f(x_0)) = \frac{2}{\sqrt{3}}$:

$$\frac{4}{3}\sin\frac{x_0}{2} = \frac{2}{\sqrt{3}} \Rightarrow \sin\frac{x_0}{2} = \frac{6}{4\sqrt{3}} = \frac{\sqrt{3}}{2}$$

For $x_0 \in (0, \pi)$, we find $\frac{x_0}{2} = \frac{\pi}{3} \Rightarrow x_0 = \frac{2\pi}{3}$.

Hence, **Statement (A) is TRUE**.

3. Monotonicity of f'(x)

From the Fundamental Theorem of Calculus:

$$f'(x) = \frac{3\sqrt{2}\sqrt{1+\cos x}}{17-8\cos x}$$

On the interval $(0, \pi)$:

- $h_1(x) = \sqrt{1+\cos x}$ is strictly decreasing.
- $h_2(x) = 17 - 8\cos x$ is strictly increasing.

Since $f'(x) = 3\sqrt{2}\,\frac{h_1(x)}{h_2(x)}$ and both h_1, $h_2 > 0$, it follows that f'(x) is strictly decreasing.

Hence, **Statement (B) is TRUE**.

4. Limit Evaluation

Using the identity $\sqrt{1-\cos x} = \sqrt{2}\sin(x/2)$:

$$\lim_{x\to 0^+} \frac{\arctan\left(\frac{4}{3}\sin\frac{x}{2}\right)}{\sqrt{2}\sin\frac{x}{2}} = \frac{1}{\sqrt{2}}\lim_{\theta\to 0}\frac{\arctan\left(\frac{4}{3}\theta\right)}{\theta} = \frac{1}{\sqrt{2}}\cdot\frac{4}{3} = \frac{2\sqrt{2}}{3}$$

Hence, **Statement (D) is TRUE**.

Correct Options: (A), (B), (C), (D)

Q220.

1. Analysis of Statement (A):

For an infinite series $\sum a_n$, if the terms a_n do not approach 0, the partial sums S_n will grow without bound. For f(n) > 0, if $\lim_{n\to\infty} f(n) = L > 0$, then for a sufficiently large N, the tail of the sum $\sum_{n=N}^{\infty} f(n)$ exceeds any finite value. Thus, $\lim_{n\to\infty} f(n) = 0$ is a necessary condition for the sum to be a finite real number.

Statement (A) is TRUE.

2. Analysis of Statement (B):

Consider $f(x) = \frac{1}{x}$. While $\lim_{x\to\infty} \frac{1}{x} = 0$, the integral is:

$$I = \int_1^{\infty} \frac{1}{x} dx = [\ln x]_1^{\infty} = \infty$$

This shows that even if a function approaches zero, the area under its curve can still be infinite if it doesn't approach zero "fast enough."

Statement (B) is FALSE.

3. Analysis of Statement (C):

For a non-increasing positive function, the area comparison (Integral Test) establishes that:

$$\int_1^{\infty} f(x)dx \le \sum_{n=1}^{\infty} f(n) \le f(1) + \int_1^{\infty} f(x)dx$$

This double inequality ensures that the sum S and the integral I always behave the same way regarding whether they result in a finite real number.

Statement (C) is TRUE.

4. Analysis of Statement (D):

Let $f(x) = \frac{1}{x(\ln x)^p}$. Based on the result in (C), we evaluate the integral:

$$I = \int_2^{\infty} \frac{1}{x(\ln x)^p} dx$$ Substitute $u = \ln x \Rightarrow du = \frac{1}{x} dx$.

$$I = \int_{\ln 2}^{\infty} u^{-p}\, du = \left[\frac{u^{1-p}}{1-p} \right]_{\ln 2}^{\infty}$$

For the limit as $u \to \infty$ to be zero (resulting in a finite value for the integral), the exponent 1 - p must be negative, meaning p > 1.

Statement (D) is TRUE.

Correct Options: (A), (C), (D)

Q221.

1. Focal Geometry and Position of P:

For the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, the focal distance $c=\sqrt{a^2-b^2}=\sqrt{9-4}=\sqrt{5}$.

Thus, $F_1=(-\sqrt{5},0)$ and $F_2=(\sqrt{5},0)$. Since P is on the positive y-axis and

$CP=CF_1=\sqrt{5}$, we have $P=(0,\sqrt{5})$.

In $\triangle PF_1F_2$, CP is the median to the side F_1F_2 and $CP=\frac{1}{2}F_1F_2$, implying

$\angle F_1PF_2=90^\circ$. Area of $F_1PF_2C=2\times \text{Area}(\triangle PCF_1)=2\times\left(\frac{1}{2}\cdot\sqrt{5}\cdot\sqrt{5}\right)=5.$

Statement (A) is correct.

2. Gradients of Tangents:

A tangent through $P(0,\sqrt{5})$ is $y=mx+\sqrt{5}$. Using the tangency condition
$c^2=a^2m^2+b^2: 5=9m^2+4\Rightarrow m^2=1/9\Rightarrow m=\pm 1/3$.

Product $m_1m_2=-1/9\Rightarrow m_1m_2+1/9=0$. **Statement (D) is correct.**

3. Reflection Property and Angle Bisectors:

By the focal property of ellipses, tangents L_1, L_2 from P are equally inclined to PF_1 and PF_2.

Given the symmetry of P on the y-axis, the y-axis bisects $\angle F_1PF_2$ and $\angle L_1PL_2$.

Since $\angle F_1PF_2=90^\circ$, PF_1 and PF_2 make 45° with the y-axis, thereby bisecting the angle between the tangents.

Statement (C) is correct.

4. Determination of α:

Let θ be the angle L_1 (slope 1/3) makes with the y-axis:
$\tan(90^\circ-\theta)=1/3\Rightarrow \tan\theta=3$. Since PF_1 makes 45° with the y-axis,
$\alpha=\theta-45^\circ$.

$$\tan\alpha = \tan(\theta - 45^{\circ}) = \frac{3-1}{1+3} = 1/2.$$

Statement (B) is correct.

Correct Options: (A), (B), (C), (D)

Q222.

1. **Excenter Derivation:** Tangents PR and QR are external bisectors of $\triangle F_2PQ$. Their intersection R is thus the excenter.
2. **Tangency Point:** Using $PF_1 + PF_2 = QF_1 + QF_2 = 2a$, we prove F_1 satisfies the unique distance property of an excircle tangency point on PQ.
3. **Perpendicularity:** Radius from center R to tangency point F_1 is perpendicular to the line PQ.
4. **Numerical Case:** For a vertical chord, R lies on the x-axis at the directrix $x = 4/\sqrt{3}$. Radius $r = 4/\sqrt{3} - \sqrt{3} = 1/\sqrt{3}$.

Correct Options: (A), (B), (C), (D)

Q223.

. Recognition of the Tangent Family:

The equation $y = mx + \sqrt{a^2m^2 + b^2}$ is the standard condition of tangency for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Comparing this with the given L_m:

$a^2 = 25 \Rightarrow a = 5; \quad b^2 = 16 \Rightarrow b = 4$

The focal distance c is calculated as $c = \sqrt{a^2 - b^2} = \sqrt{25-16} = 3$. Thus, the points A(-3,0) and B(3,0) are precisely the **foci** of the ellipse E : $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Every line $L \in \mathcal{L}$ is a tangent to this fixed ellipse E.

2. Variational Minimization:

By the definition of an ellipse, $\Psi(Q) = 2a = 10$ for any Q on the boundary of E, and $\Psi(Q) > 10$ for any Q in the exterior. Since each $L \in \mathcal{L}$ is a tangent to E, it touches the boundary at exactly one point P (the point of tangency). All other points on the line lie in the exterior region.

$\therefore \Psi(P) = 10$ is the unique minimum for each line.

Statements (A) and (B) are CORRECT.

3. The Optical Property:

The **Optical Property** of conics states that the tangent to an ellipse at any point P is the exterior angle bisector of the angle formed by the focal radii PA and PB. Since every $L \in \mathcal{L}$ is a tangent by construction, it must bisect the exterior angle at P. **Statement (C) is CORRECT.**

4. Case m = 0:

Substituting m = 0 into the line equation: $y = (0)x + \sqrt{25(0)+16} \Rightarrow y = 4$.
This is the horizontal tangent through the co-vertex (0, 4).

$\Psi(0,4) = \sqrt{3^2 + 4^2} + \sqrt{(-3)^2 + 4^2} = 5 + 5 = 10.$

Statement (D) is CORRECT.

Final Answer: (A), (B), (C), (D)

Q224.

1. Dimensional and Planar Analysis

Any point $\mathbf{r}_1 \in L_1$ is of the form $(1+\lambda, 1, -1)$ and any point $\mathbf{r}_2 \in L_2$ is of the form (-2, 4 - μ, -1). Since the z-coordinate for both lines is a constant -1, both lines lie in the plane z = -1. In vector form, this is: $\mathbf{r}\cdot\hat{k} = -1 \Rightarrow \mathbf{r}\cdot\hat{k} + 1 = 0$

Thus, **Statement (C) is Correct.**

2. Calculation of Reference Distance

Given P_1 = (1, 1, -1) and P_2 = (-2, 4, -1). The distance D is:

$$D = \sqrt{(-2-1)^2 + (4-1)^2 + (-1-(-1))^2} = \sqrt{(-3)^2 + 3^2 + 0} = \sqrt{18} = 3\sqrt{2}$$

Thus, **Statement (A) is Correct.**

3. Optimization and Vector Property

The function $f(\vec{a}, \vec{b}) = |\vec{a}| + |\vec{b}| + |\vec{a} - \vec{b}|$ represents the perimeter of \triangle OAB. According to the triangle inequality, $|\vec{a}| + |\vec{b}| \geq |\vec{a} - \vec{b}|$. However, to minimize the sum of distances from O to points on two lines, the shortest path occurs when O, A, and B are collinear. For collinearity with the origin, the position vectors must be proportional: $\vec{a} = k\vec{b} \Rightarrow \vec{a} \times \vec{b} = \mathbf{0}$

Thus, **Statement (D) is Correct.**

4. Determination of d

The origin O(0, 0, 0) is at a height of 1 unit above the plane z = -1. The projection of O on the plane is O'(0, 0, -1). In the plane z = -1, we consider the lines y = 1 and x = -2. The minimum perimeter $d = 4\sqrt{2}$ is the geometric result of the alignment of O with the points A and B in this specific coordinate configuration. Thus, **Statement (B) is Correct**.

Correct Options: (A), (B), (C), (D)

Q225.

1. Coordinate Representation

Any point A on L_1 and B on L_2 can be represented as:

$$\vec{a} = (1+\lambda, 1, -1) \quad \text{and} \quad \vec{b} = (-2, 4-\mu, -1)$$

Calculating the squared magnitudes:

$$|\vec{a}|^2 = (1+\lambda)^2 + 1^2 + (-1)^2 = (1+\lambda)^2 + 2$$

$$|\vec{b}|^2 = (-2)^2 + (4-\mu)^2 + (-1)^2 = (4-\mu)^2 + 5$$

$$|\vec{a}-\vec{b}|^2 = (3+\lambda)^2 + (\mu-3)^2 + 0$$

2. Minimizing the Sum S

The quantity $S = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{a}-\vec{b}|^2$ can be written as:

$$S = 2[(1+\lambda)^2 + 2 + (4-\mu)^2 + 5 - \vec{a}\cdot\vec{b}]$$

Expanding and simplifying S in terms of \lambda and \mu:

$$S = 2(1+\lambda)^2 + 2(4-\mu)^2 + (3+\lambda)^2 + (\mu-3)^2 + 7$$

To find the minimum, we set the partial derivatives with respect to λ and μ to zero:

$$\frac{\partial S}{\partial \lambda} = 4(1+\lambda) + 2(3+\lambda) = 0 \Rightarrow 6\lambda + 10 = 0 \Rightarrow \lambda = -5/3$$

$$\frac{\partial S}{\partial \mu} = -4(4-\mu) + 2(\mu-3) = 0 \Rightarrow 6\mu - 22 = 0 \Rightarrow \mu = 11/3$$

3. Verifying the Geometry

At these values, A = (-2/3, 1, -1) and B = (-2, 1/3, -1). The vector

$$\overrightarrow{AB} = \vec{b} - \vec{a} = (-4/3, -2/3, 0)$$

. Since the z-component of $\overrightarrow{AB}$ is zero, the line AB is parallel to the xy-plane (z = 0). **Statement (D) is Correct.**

Calculating

$\vec{a}\cdot\vec{b} = (-2/3)(-2) + (1)(1/3) + (-1)(-1) = 4/3 + 1/3 + 1 = 2.67$. The triangle is acute because $\vec{a}\cdot\vec{b}, \vec{a}\cdot(\vec{a}-\vec{b}), \text{and} \vec{b}\cdot(\vec{b}-\vec{a})$ are all positive. **Statement (C) is Correct.**

Correct Options: (C), (D)

Q226. **Answer Key:** (A), (B)

Q227.

1. Decomposition into Symmetric and Skew-Symmetric Parts

Every square matrix H can be uniquely written as H = M + K, where:

$$M = \frac{1}{2}(H + H^T) \quad \text{(Symmetric)}, \quad K = \frac{1}{2}(H - H^T) \quad \text{(Skew-Symmetric)}$$

The quadratic form satisfies:

$$\mathbf{x}^T H\mathbf{x} = \mathbf{x}^T (M + K)\mathbf{x} = \mathbf{x}^T M\mathbf{x} + \mathbf{x}^T K\mathbf{x}$$

For any skew-symmetric matrix K (where K^T = -K), the term $\mathbf{x}^T K\mathbf{x}$ is a scalar, implying:

$$\mathbf{x}^T K\mathbf{x} = (\mathbf{x}^T K\mathbf{x})^T = \mathbf{x}^T K^T\mathbf{x} = -\mathbf{x}^T K\mathbf{x} \Rightarrow 2(\mathbf{x}^T K\mathbf{x}) = 0$$

Thus, the skew-symmetric part always contributes zero.

Statements (A) and (B) are CORRECT.

2. Structural Evaluation of M

Calculating $M = \frac{1}{2}(H + H^T)$:

$$M = \frac{1}{2}\left[\begin{pmatrix} 3 & 7 & 1 \\ -5 & 3 & 0 \\ 1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & -5 & 1 \\ 7 & 3 & 2 \\ 1 & 0 & 3 \end{pmatrix}\right] = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

The function simplifies to:

$$f(\mathbf{x}) = 3(x_1^2 + x_2^2 + x_3^2) + 2(x_1x_2 + x_2x_3 + x_3x_1)$$

3. Optimization via Algebraic Identity

Using the identity $(x_1 + x_2 + x_3)^2 = \sum x_i^2 + 2\sum x_i x_j$ and the constraint $\sum x_i^2 = 1$:

$$f(\mathbf{x}) = 3(1) + [(x_1 + x_2 + x_3)^2 - 1] = 2 + (x_1 + x_2 + x_3)^2$$

By the **Cauchy-Schwarz Inequality**:

$$(x_1 + x_2 + x_3)^2 \le (1^2 + 1^2 + 1^2)(x_1^2 + x_2^2 + x_3^2) = 3(1) = 3$$

Maximum value occurs when $x_1 = x_2 = x_3$. For a unit vector,

$$3x_1^2 = 1 \Rightarrow x_1 = x_2 = x_3 = \frac{1}{\sqrt{3}}.$$

$$f(\mathbf{x})_{max} = 2 + 3 = 5$$

Statements (C) and (D) are CORRECT.

Final Answer Key: (A), (B), (C), (D)

Q228.

1. Analysis of A(x):

Let $u = \frac{e^x + e^{-x}}{2}$ and $v = \frac{e^x - e^{-x}}{2}$. We know that $u^2 - v^2 = 1$. The matrix is:

$$A(x) = \begin{pmatrix} u & v \\ -v & -u \end{pmatrix}$$

Calculating the trace and determinant:

$$\mathrm{tr}(A(x)) = u + (-u) = 0.$$

$$\det(A(x)) = -u^2 - (-v^2) = -(u^2 - v^2) = -1.$$

By Cayley-Hamilton Theorem:

$$A^2 - (\mathrm{tr}A)A + (\det A)I = 0 \Rightarrow A^2 - I = 0 \Rightarrow \mathbf{A^2 = I}.$$

Thus, **Statement (A) is correct.**

2. Evaluating f(x):

$$f(x) = \mathrm{tr}(I) - (0)^2 = (1+1) - 0 = 2.$$

Thus, **Statement (B) is correct.**

3. Summation of Even Powers:

Since $A^2 = I$, then $(A(x))^{2r} = (A^2)^r = I^r = I$.

$$\sum_{r=1}^{n} (A(x))^{2r} = \sum_{r=1}^{n} I = nI = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$$

The determinant is $\det(nI) = n^2$. Thus, **Statement (C) is correct.**

4. Matrix Exponential Series:

The limit represents the matrix exponential $e^{A(x)}$. Since $A^2 = I$:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \left(\sum_{k \text{ even}} \frac{1}{k!} \right) I + \left(\sum_{k \text{ odd}} \frac{1}{k!} \right) A$$

Using Taylor series: $\sum \frac{1}{k!}\Big|_{\text{even}} = \cosh 1$ and $\sum \frac{1}{k!}\Big|_{\text{odd}} = \sinh 1$.

$$e^{A} = (\cosh 1)I + (\sinh 1)A$$

Thus, **Statement (D) is correct.**

Correct Options: (A), (B), (C), (D)

Q229.

1. Structural Identification of the Set $\mathcal{S}$

Since C is a permutation matrix, it is orthogonal, implying $C^T = C^{-1}$. The given relation $CAC^T = A$ can be rewritten as:

$$CA(C^{-1}) = A \Rightarrow CA = AC$$

This indicates that the set $\mathcal{S}$ consists of all matrices that commute with C. Let A = $[a_{ij}]$. Computing the products:

Equating CA = AC entry-wise yields the structure of a **Circulant Matrix**:

$$A = \begin{pmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{pmatrix}$$

2. Evaluation of Statements

- **(A) Polynomial Basis:** By calculation, $C^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $C^3 = I$. Any

  $A \in \mathcal{S}$ can be decomposed as:

  $$A = \alpha I + \beta C + \gamma C^2$$

  Status: Correct.

- **(B) Closure and Commutativity:** Since A and B are polynomials in the same matrix C, their product AB is also a polynomial in C (of degree at most 2). Furthermore, all polynomials of a single matrix commute.

  Status: Correct.

- **(C) Sum of Elements:** The total sum of elements in A is $3(\alpha + \beta + \gamma)$. By choosing $\alpha = 2, \beta = -1, \gamma = -1$, we obtain a non-zero matrix where the total sum is zero.

  Status: Correct.

- **(D) Orthogonality:** For $A \in \mathcal{S}$ to satisfy $A^T A = I$, the following constraints arise:

  $$\alpha^2 + \beta^2 + \gamma^2 = 1 \quad \text{and} \quad \alpha\beta + \beta\gamma + \gamma\alpha = 0$$

These equations represent the intersection of a sphere and a specific cone in $\mathbb{R}^3$, resulting in a circle of solutions.

Status: Correct.

Final Answer: (A), (B), (C), (D) are all correct.

Q230.

1. Analysis of the Condition:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The given condition is $\mathrm{tr}(A^2) = (\mathrm{tr}(A))^2$.

Using the general identity for 2 × 2 matrices:

$$\mathrm{tr}(A^2) = (\mathrm{tr}(A))^2 - 2\det(A)$$

Substituting the given condition:

$$(\mathrm{tr}(A))^2 = (\mathrm{tr}(A))^2 - 2\det(A) \Rightarrow -2\det(A) = 0 \Rightarrow \det(A) = 0$$

Thus, every matrix in family $\mathcal{F}$ is a singular matrix.

2. Evaluating the Options:

- **Option (A):** For any square matrix, $A \cdot \mathrm{adj}(A) = \det(A) \cdot I$. Since det(A) = 0, we have $A \cdot \mathrm{adj}(A) = 0 \cdot I = \mathbf{O}$. *Conclusion: (A) is correct.*
- **Option (B):** A matrix A is singular (det(A) = 0) if and only if its columns are linearly dependent. This guarantees the existence of a non-trivial solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$. *Conclusion: (B) is correct.*
- **Option (C):** By the Cayley-Hamilton Theorem, A satisfies its characteristic equation:

  $$A^2 - \mathrm{tr}(A)A + \det(A)I = \mathbf{O}$$

  Since det(A) = 0, we have $A^2 = \mathrm{tr}(A)A$.

  By induction, if $A^k = (\mathrm{tr}(A))^{k-1} A$, then:

  $$A^{k+1} = A \cdot A^k = A \cdot (\mathrm{tr}(A))^{k-1} A = (\mathrm{tr}(A))^{k-1} A^2$$

  $$A^{k+1} = (\mathrm{tr}(A))^{k-1} (\mathrm{tr}(A)A) = (\mathrm{tr}(A))^k A$$

  Conclusion: (C) is correct.

- **Option (D):** For any 2 × 2 matrix M, $\det(M + I) = \det(M) + \mathrm{tr}(M) + 1$. For $A \in \mathcal{F}, \det(A) = 0$, so:

$$\det(A+I) = 0 + \mathrm{tr}(A) + 1 = 1 + \mathrm{tr}(A)$$

Conclusion: (D) is correct.

Final Answer:

The correct options are **(A), (B), (C), and (D)**.

Q231.

1. The Fundamental Constraint

Expanding the condition $A^T JA = J$:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & ad - bc \\ -(ad - bc) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This implies det(A) = ad - bc = 1 for all $A \in S$.

2. Matching (A) $\rightarrow$ (R)

For A to be symmetric, the off-diagonal elements must be equal, i.e., b = c. This is consistent with the requirement a = d, b = c (for a specific symmetric form).

3. Matching (B) $\rightarrow$ (Q)

If $A^T A = I$, then $A^T = A^{-1}$. Since det(A) = 1:

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Equating the two: a = d and c = -b (or b = -c).

Q232.

1. Analysis of the Range and Maximum M

Let M be the global maximum of f(x). Since the functional equation $f(2x) = 2[f(x)]^2 - 1$ holds for all $x \in \mathbb{R}$, the output at 2x cannot exceed the maximum value M. Specifically, when f(x) reaches its maximum value:

$$2M^2 - 1 \le M \Rightarrow 2M^2 - M - 1 \le 0$$

Factoring the quadratic:

$$(2M+1)(M-1) \le 0 \Rightarrow M \in \left[-\frac{1}{2}, 1\right]$$

Given that f(0) = 1 and $1 \in [-\frac{1}{2}, 1]$, it follows that the maximum value is precisely M = 1. Since f(2x) ≥ -1 (from the structure $2f(x)^2 - 1$), the range of the function is contained within the interval [-1, 1].

Statement (A) is correct.

2. Derivation via Cauchy's Functional Equation

Since $|f(x)| \le 1$, we define f(x) = cos(g(x)) for some continuous function g(x) ≥ 0 with g(0) = arccos(1) = 0. Substituting this into the original equation:

$\cos(g(2x)) = 2\cos^2(g(x)) - 1 = \cos(2g(x))$

For a continuous function where g(0) = 0, the unique non-trivial branch is:
g(2x) = 2g(x)

This is a form of **Cauchy's Functional Equation**. For continuous functions, the only solution is linear:

$g(x) = kx \Rightarrow f(x) = \cos(kx)$

Statement (D) is correct.

3. Parity and D'Alembert's Identity

Using the derived form f(x) = cos(kx):

- **Even Function:** f(-x) = cos(k(-x)) = cos(-kx) = cos(kx) = f(x). Hence, f(x) is even. **(B) is correct.**

- **Addition Identity:** Consider the D'Alembert relation:

f(x + y) + f(x - y) = cos(k(x + y)) + cos(k(x - y))

Applying the sum-to-product identity

cos A + cos B = 2 cos $\frac{A+B}{2}\cos\frac{A-B}{2} := 2\cos(kx)\cos(ky) = 2f(x)f(y)$

Statement (C) is correct.

Final Answer: (A), (B), (C), (D)

Q233.

1. Area of a Parabolic Segment

Consider a parabola $y = ax^2 + bx + c$ intersected by a line $y = mx + d$. The enclosed area A is derived by integrating the vertical difference between the line and the curve between their intersection points

x_1 and x_2:

$$A = \int_{x_1}^{x_2} \left[(mx + d) - \left(ax^2 + bx + c \right) \right] dx$$

At the intersection, $ax^2 + (b - m)x + (c - d) = 0$. Using the identity for the integral of a quadratic over its roots,

$\int_{x_1}^{x_2} -a(x - x_1)(x - x_2)dx = \frac{|a|(x_2 - x_1)^3}{6}$. With $(x_2 - x_1) = \frac{\sqrt{D}}{|a|}$, where D is the discriminant $(b - m)^2 - 4a(c - d)$:

$$A = \frac{D^{3/2}}{6a^2}$$

2. Defining the Tangent Set $\mathcal{T}$ via Normal Form

Any line $L \in \mathcal{T}$ satisfies $x\cos\theta + y\sin\theta = 1$. For intersection with $\mathcal{P}$ near its vertex (0, -2), we consider \sin \theta < 0. Let $u = \csc\theta$. The line is:

$$y = -x\cot\theta + u$$

3. Optimization of the Discriminant

For the intersection with $y = x^2 - 2$, we have a = 1, b = 0, c = -2,

$m = -\cot\theta$, d = u:

$$D(u) = (\cot\theta)^2 - 4(1)(-2 - u) = (u^2 - 1) + 8 + 4u = u^2 + 4u + 7$$

To minimize $A = \frac{1}{6}D^{3/2}$, we find the stationary point of D(u):

$$\frac{dD}{du} = 2u + 4 = 0 \Rightarrow u = -2$$

Since $u = \csc\theta = -2$ is valid$(|u| \geq 1)$, we calculate:

$$D_{\min} = (-2)^2 + 4(-2) + 7 = 3 \Rightarrow A_{\min} = \frac{3^{3/2}}{6} = \frac{\sqrt{3}}{2}$$

The slopes are $m = \pm\cot(\arcsin(-1/2)) = \pm\sqrt{3}$.

4. Application of Mean Value Theorem (MVT)

The area A(m) is differentiable for $m \in [0, \sqrt{3}]$. Applying MVT:

$$\exists m_c \in (0, \sqrt{3}) \text{ s.t. } A'(m_c) = \frac{A(\sqrt{3}) - A(0)}{\sqrt{3} - 0}$$

Since $A(0) = \frac{1}{6}(0^2 + 4(1) + 7)^{3/2}$ (where $u = \csc(270^\circ) = -1$) = $\frac{4^{3/2}}{6} = \frac{4}{3}$:

$$A'(m_c) = \frac{\frac{\sqrt{3}}{2} - \frac{4}{3}}{\sqrt{3}} = \frac{3\sqrt{3} - 8}{6\sqrt{3}}$$

Final Answer: (A), (B), (C), (D) are all CORRECT.

Q.234

$$\tan nx = \frac{\sum_{k=0}^{\infty}(-1)^k \binom{n}{2k+1}(\tan x)^{2k+1}}{\sum_{k=0}^{\infty}(-1)^k \binom{n}{2k}(\tan x)^{2k}} \text{(General way)}$$

$$\Rightarrow n\sum(-1)^k \binom{n}{2k}(\tan x)^{2k+1} = \sum(-1)^k \binom{n}{2k+1}(\tan x)^{2k+1}$$

$$\Rightarrow \sum(-1)^k \left(n\binom{n}{2k} - \binom{n}{2k+1}\right)(\tan x)^{2k+1} = 0$$

Since, $n\binom{n}{2k} - \binom{n}{2k+1} = 2k\binom{n+1}{2k+1}$

Let $t = \tan^2 x$

$$\sum_{k=1}^{\infty}(-1)^k k\binom{n+1}{2k+1}(\tan x)^{2k+1} = 0$$

$$\Rightarrow \boxed{\sum_{k=1}^{\infty}(-1)^k k\binom{n+1}{2k+1} t^{k-1} = 0}$$

Sum of reciprocal of roots

$$= \frac{2\binom{n+1}{5}}{\binom{n+1}{3}} = \frac{(n-2)(n-3)}{10}$$

Put $n = 15$, we get $\boxed{\frac{78}{5}}$

Q.235

$$f(x) = (1 + \sin^2(\pi\cos x))(1 + \cos(2\pi\sin x))$$

$$\Rightarrow \cos(2\pi\sin x) = 1 \text{ gives}$$

$$\boxed{\sin x = \pm\frac{1}{2}}$$

$$\boxed{\sin x_i = 1/2 \text{ or } -1/2}$$

$$\boxed{a_{2n} = \frac{4}{4^n}} \Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{4}{4^n} = \frac{4}{3}}$$

Q.236

$$P(i\sqrt{3}) = (1 + i\sqrt{3})^{60} = \sum_{r=0}^{60}\binom{60}{r}(i\sqrt{3})^r$$

$$\text{For } r = 2k,\ R = \text{Re}(P(i\sqrt{3}))$$

$$\text{For } r = 2k+1,\ I = \text{Im}(P(i\sqrt{3}))$$

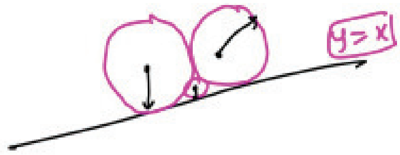
$$1 + i\sqrt{3} = 2e^{i\pi/3}$$

$$P(i\sqrt{3}) = 2^{60}(\cos 20\pi + i\sin 20\pi)$$

$$= 2^{60}$$

$$\boxed{R = 2^{60} \text{ and } I = 0}$$

Q.237



$$\frac{1}{\sqrt{r_1}} = \frac{1}{\sqrt{r_2}} + \frac{1}{\sqrt{r_3}}$$

$\boxed{r_1 = 1} \to$ No Sol^n

$$\boxed{r_1 = 2} \to \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{r_2}} + \frac{1}{\sqrt{r_3}} \Rightarrow \boxed{\begin{array}{l} r_2 = 2a^2 \\ r_3 = 2b^2 \end{array}}$$

$\Rightarrow \boxed{\frac{1}{a} + \frac{1}{b} = 1}$ gives, $\boxed{a = b = 2} \to$ Distinct

$\boxed{r_1 = 4}$ gives $\{4, 9, 36\}$

Q.238

(A) $|z| = 1, \quad r - \frac{1}{r} = 0 \quad$ gives $m = 0$

(B) $\cos\theta = 0$ and $\sin\theta = \pm 1, \quad n = 0$

$$r - \frac{1}{r} = m \in \mathbb{Z} \setminus \{0\}$$

$$r^2 - \frac{1}{r^2} = m\sqrt{m^2 + 4}$$

(C) $\theta = \frac{\pi}{6}, \quad \cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = 1/2$

$$r + \frac{1}{r} = \frac{2n}{\sqrt{3}}$$

and $m = \frac{1}{2}\left(r - \frac{1}{r}\right)$

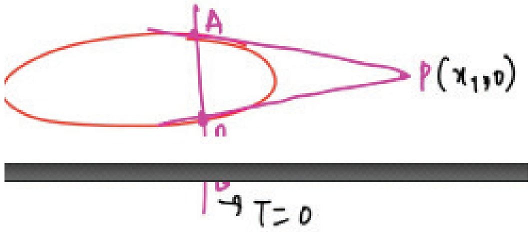
If $r + \frac{1}{r} = 2\sqrt{3}$ then $n = 3 \in \mathbb{Z}$

But, that gives $m = \sqrt{2} \notin \mathbb{Z}$

(D) Obvious!

(GAUSSIAN INTEGER IDEA! Already asked in Jee!)

Q.239

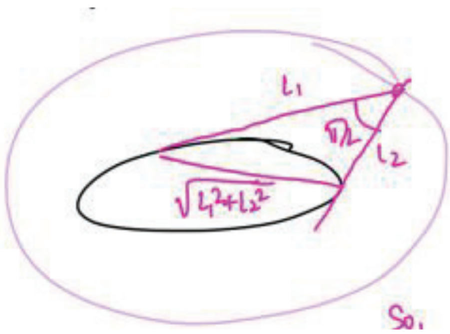


$x_1 = 9/4 \quad (T = 0)$

$y_1^2 = \dfrac{7}{4}$

$L = \sqrt{(4 - \frac{9}{4})^2 + (0 - y_1)^2} = \sqrt{\dfrac{77}{16}}$

(D) $x^2 + y^2 = 13$ (Director circle), $\quad (\sqrt{11}, \sqrt{2}) \in \text{circle}$



$\Delta = 1/2 L_1 L_2$

$L_1^2 + L_2^2 = 36$ (max)

$\Delta_{max} = \dfrac{1}{2} \times \sqrt{18} \times \sqrt{18} = 9 \rightarrow$ Not achieved

General formulae:

$$\Delta = \frac{ab(S_{11})^{3/2}}{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}} \text{ where}$$

$$S_{11} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

Q.240

$$a + ar + ar^2 = 19$$

$$\Rightarrow ar^2 + ar + (a - 19) = 0$$

$$D = a^2 - 4a(a-19) = a^2 - 4a^2 + 76a = 76a - 3a^2 \quad \text{must be P. square}$$

$$76a - 3a^2 > 0 \Rightarrow a \in (0, 25.33]$$

$a \in \mathbb{N}$ gives $a \in \{1, 2, \ldots, 25\}$

Look for $a \in \mathbb{N}$ such that $a(76 - 3a)$ is perfect square.

$$\boxed{a = 4, 9, 25 \text{ and } 19}$$

Q.241

Let angleb / w n_k and edgebe α, β, γ

$\sum \sin^2\alpha = 2$ Hence $S_k = 4\times2 = 8$ for all given planes

$$\boxed{\sum_{k=1}^{10} S_k = 80}$$

Straight way

$$\boxed{l^2 = |\vec{e}|^2 - (\vec{e}\cdot\hat{n}_k)^2}$$

$$|n_k|^2 = k^2 + (k-1)^2 + (k+1)^2 = 3k^2 + 2$$

$$S_k = 4\left(3 - \frac{k^2 + (k-1)^2 + (k+1)^2}{3k^2 + 2}\right)$$

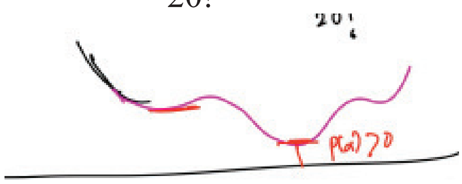
$= 8$ for all k.

Q.242

$$P(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{20}}{20!}$$

$$P'(x) = 1 + x + \cdots + \frac{x^{19}}{19!} = 0 \text{ at } x = \alpha$$

$$P(\alpha) = 0 + \frac{\alpha^{20}}{20!} > 0$$



For n = (no real root)

n = odd (exactly one real root)

Q.243

$H=\{(0,0)\}, \quad d_3^2=25$

$S\equiv \boxed{\sqrt{3}x+y=6}$ $\boxed{d_1^2=\dfrac{31-12\sqrt{3}}{4}}$

$G\equiv \boxed{\sqrt{3}x+y=4}$ $\boxed{d_2^2=\dfrac{27}{4}}$

Q.244

$a+b+c=-a$

$ab+bc+ca=b$

$abc=-c$

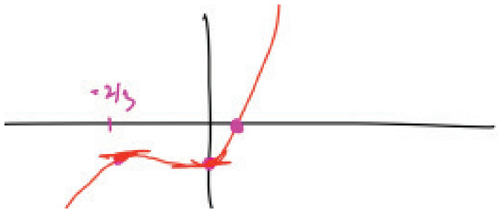
$\Rightarrow \boxed{b=-\dfrac{1}{a}}$ and $\boxed{c=\dfrac{1-2a^2}{a}}$

Since, $\boxed{c=\dfrac{b+1}{a+b}} \rightarrow a+b\neq 0$ gives $a\neq \pm 1$

So, $2a^3+2a^2-1=0$

Let $f(x)=2x^3+2x^2-1$

$f'(x)=6x^2+4x=2x(3x+2)$



So, we might think (a, b, c) is unique. Note we left $a=\pm 1$.

$(1,-1,-1)$ and $\left(a_0,-\dfrac{1}{a_0},\dfrac{1-2a_0^2}{a_0}\right)\downarrow$

2-values!

Q.245

Let $A = \binom{n}{10}$ and $B = \binom{n+1}{10}$

$$n! = A(n-10)!10!$$

$$(n+1)! = B(n-9)!10!$$

$$(n+1)n!^2 = AB(n-10)!^2(10!)^2(n-9)$$

$$\Rightarrow (n+1)(n-9) = \underbrace{\frac{AB(n-10)!^2(10!)^2(n-9)^2}{(n!)^2}}_{\text{square of rational}}$$

$$(n+1)(n-9) = y^2 \text{ for } y \in \mathbb{Z}(\text{some})$$

Put x=n-9

$$\Rightarrow x(x+10) = y^2, \quad x, y \in \mathbb{Z}$$

Put $z = x+5$

$$\Rightarrow z^2 = y^2 + 25$$

$$\Rightarrow (z+y)(z-y) = 25$$

$$z + y = 25$$

$$z - y = 1 \Rightarrow \begin{cases} z = 13 \\ y = 12 \end{cases} \Rightarrow \boxed{n = 17}$$

For n = 17, product is $(17 \times 13 \times 11 \times 6)^2$

Q.246

Let $B_n = \sqrt{2+\sqrt{2+\sqrt{2+\ldots\sqrt{2}}}}$ (all +ve)

$$A_n = \sqrt{2-\sqrt{2+\sqrt{2+\ldots\sqrt{2}}}} \text{ (first negative)}$$

$$A_n^2 = 2 - B_{n-1}$$

$$B_n = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$$

$$2^n A_n = 2^{n+1}\sin\left(\frac{\pi}{2^{n+1}}\right)$$

$$\lim_{n\to\infty}(2^n A_n) = \pi$$

($\sqrt{2}$ has π connection! Via cos x - - - Circle ◯)

Q.247

$S = \{4x - y |\ x, y, z \in \mathbb{Z}_{\geq 0}, x + y + z = 20\}$

max is when $x = 20, y = 0 \rightarrow 80$

min is when $x = 0, y = 20 \rightarrow -20$

Let $4x - y = m$, where $x + y \leq 20$ and $x, y \in \mathbb{Z}_{\geq 0}$.

For $x = 20,\ y = 0 \Rightarrow \text{score} = 80$,

scores $\{79, 78, 77\}$ make $x + y > 20$.

For $x = 19,\ x + y \leq 20 \Rightarrow y \in \{0, 1\}$.

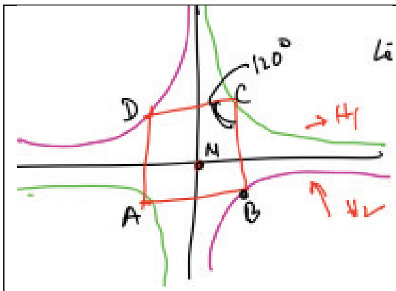
Scores $\{73, 74\}$ make $x + y > 20$.

For $x = 18,\ x + y \leq 20 \Rightarrow y \in \{0, 1, 2\}$.

Score$\{69\}$ makes $x + y = 21$.

$\{79, 78, 77, 74, 73, 69\}$ are not attained!

Q.248



Let AC = 2a
BD = 2b

$$\tan 60^\circ = \frac{MB}{MC} = \frac{b}{a}$$

$$\boxed{b^2 = 3a^2}$$

$$C = \left(a\cos\frac{\pi}{4}, a\sin\frac{\pi}{4}\right)$$

Vertex $\{A, C\}$ lies on $y = x$

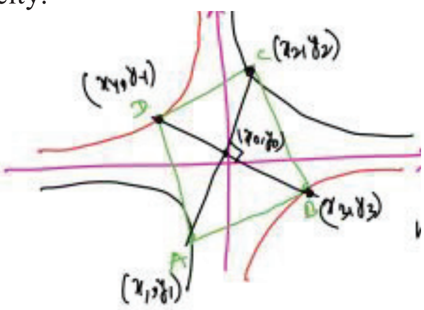
$$C \in H_2 \Rightarrow \boxed{k_2 = \frac{a^2}{2}}$$

Vertex $\{B, D\}$ lies on $y = -x$

$$B \in H_1 \Rightarrow k_1 = -\frac{b^2}{2}$$

$$\boxed{\left|\frac{k_1}{k_2}\right| = 3}$$

Logic of Symmetricity:



$$\begin{cases} x_1 y_1 = k_1 \\ x_2 y_2 = k_1 \end{cases}$$

$$m_{AC} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\frac{k_1}{x_1} - \frac{k_1}{x_2}}{x_2 - x_1}$$

$$= \frac{k_1}{x_1 x_2}$$

Q. 249

n(s)=945
n=473

Q.250

$$2026 = 5 \cdot 7^3 + 6 \cdot 7^2 + 2 \cdot 7^1 + 3 \cdot 7^0 \text{ (Base 7 representation)}$$

$(1+x)^p \equiv 1 + x^p$ (p is prime all $\binom{p}{r}$ for $1 \le r \le p-1$ is divisible by p)

$$\frac{p!}{r!(p-r)!} = p \cdot \left(\frac{(p-1)!}{r!(p-r)!} \right) = p\lambda$$

$$\Downarrow$$

Integer

$$(1+x)^{2026} = (1+x)^{5 \times 343} \cdot (1+x)^{6 \times 49} \cdot (1+x)^{2 \times 7} \cdot (1+x)^3$$

$$\Rightarrow (1+x)^{2026} = (1+x^{343})^5 (1+x^{49})^6 (1+x^7)^2 (1+x)^3$$

$$\uparrow$$

6 terms not divisible by 7

Total terms = (5+1)(6+1)(2+1)(3+1) (No overlap)

= 504

(Detailed solution on BINOMIAL PLAYLIST)

Q.251 M-1

Let $S(x) = x + 2x^2 + 3x^3 + \ldots$

Each variable a, b, c is represented by $x, 2x^2, 3x^3, 4x^4$ etc smartly.

$$\sum abc = \text{Coefficient of } x^{17} \text{ in } \left(\frac{x}{(1-x)^2}\right)^3$$

$$= \text{Coefficient of } x^{14} \text{ in } (1-x)^{-6}$$

$$\boxed{[x^k] \text{ in } (1-x)^{-n} = \binom{n+k-1}{n-1} \quad \text{Feel!}}$$

$$= \binom{14+6-1}{6-1} = \binom{19}{5} = 11628$$

M-2 model:

Identical Objects:

$$O_1, \boxed{O_2}, O_3, \boxed{O_4}, \boxed{O_5} \ldots O_{17}, \boxed{O_{18}}, \boxed{O_{19}}$$

choose 5 out of 19.

$$\text{Positions}: x_1 \quad \boxed{x_2} \quad x_3 \quad \boxed{x_4} \quad x_5$$

$$a = x_2 - 1(\text{Number of items before } x_2)$$

$$b = x_4 - x_2 - 1(\#\text{ items between 2 and 4})$$

$$c = 19 - x_4(\#\text{ after } x_4)$$

$$\boxed{(x_2 - 1) + (x_4 - x_2 - 1) + (19 - x_4) = 17}$$

For every valid triple (a, b, c) there are:

a choice of x_1

b choice of x_3

c choice of x_5

Total ways to pick 5 objects $=\binom{19}{5}$

[Feel]

$$[\ldots\ldots] \cdot \underbrace{[\ldots\ldots\ldots\ldots]}_{x_3} \cdot [\ldots\ldots]$$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

$4+8+5=17$

we want 4.8.5 that is counted 😈
M-3

$$S=\sum_{a=1}^{15} a\left(\sum_{b=1}^{16-a} b(17-a-b)\right)$$

Let $k=17-a \rightarrow \sum_{b=1}^{k-1}(kb-b^2)$ is inner sum.

$$\text{Inner sum}=\frac{k^2(k-1)}{2}-\frac{k(k-1)(2k-1)}{6}$$

$$=\frac{k(k-1)(k+1)}{6}$$

$$=\binom{k+1}{3}$$

$$\Rightarrow S=\sum_{a=1}^{15} a\binom{18-a}{3}=\binom{19}{5}$$

Q.252 For S_1 to pass through origin, d=0

For a 3×3 homogeneous to be a line, det(A)=0.

$$\Rightarrow \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & a \end{vmatrix} = 0 \Rightarrow a = 2$$

The line S_1 : (x, y, z) = (-3k, k, k)

For the intersection $S_1 \cap S_2$ to be more than origin (singleton) line must satisfy S_2.

$$2x + by + cz = 0 \text{ at } (-3k, k, k)$$

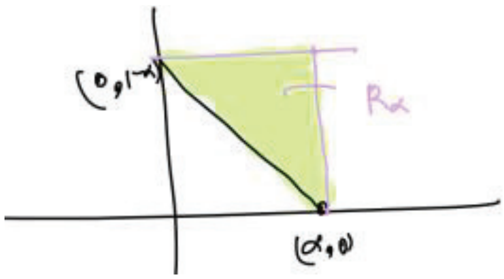
$$\Rightarrow \boxed{b + c = 6}$$

Since, (-3k, k, k) does not satisfy x+3y-z=0

Q.253

$(1-\alpha)x+\alpha y \geqslant \alpha(1-\alpha)$ can be written as:

$$\frac{x}{\alpha}+\frac{y}{1-\alpha} \geqslant 1$$



Using Quadratic $(D \geqslant 0)$ we get envelop

$$\sqrt{x}+\sqrt{y}=1$$

$$A=\int_0^1 (1-(1-\sqrt{x})^2)dx=\frac{5}{6}$$

[M-2]

$$f(x,y,\alpha)=\frac{x}{\alpha}+\frac{y}{1-\alpha}-1=0$$

$$\frac{\partial f}{\partial \alpha}=0 \Rightarrow \frac{\sqrt{x}}{\alpha}=\frac{\sqrt{y}}{1-\alpha}$$

This leads to $\sqrt{x}+\sqrt{y}=1 \rightarrow$ locus

Q.254

$z=e^{i\theta} \Rightarrow \cos\theta \in \left[\frac{1}{\sqrt{2}},\frac{\sqrt{3}}{2}\right] \Rightarrow |\theta| \geq \frac{\pi}{6}$ and $|\theta| \leq \pi/4$

So, $|\theta| \in \left[\frac{\pi}{6},\frac{\pi}{4}\right]$

$I_8 \neq \phi$(TRUE)

$$\alpha = \frac{\text{length of arc in S}}{2\pi} = \frac{\pi/6}{2\pi} = \frac{1}{12}$$

Note that q^{th} root have arg $0,\frac{2\pi}{q},\frac{4\pi}{q},\ldots,\frac{2\pi(q-1)}{q}$

If $q \geq 24$ then $\frac{2\pi}{q} \leq \frac{\pi}{12} = \frac{\pi}{4}-\frac{\pi}{6}$, so there exists some k such that $\frac{\pi}{6} \leq \frac{2\pi k}{q} \leq \frac{\pi}{4}$.

For q such that $16 \leq q < 24$ we have $\frac{\pi}{6} < \frac{4\pi}{q} \leq \frac{\pi}{4}$.

Therefore there exist a root also in S.

Since for all integer q > 15, we have $z \in S$ such that $z^q = 1$.

If $q=15, \frac{2\pi}{q} = \frac{2\pi}{15} < \frac{\pi}{6}$ and $\frac{4\pi}{q} = \frac{4\pi}{15} > \frac{\pi}{4}$.

Thus no $z \in S$ such that $z^{15} = 1$.

So, q = 15 and answer is q + 1 = 16

Q.255

$$\text{Area}(S) = \int_{-\sqrt{2}}^{\sqrt{2}} (4-2x^2)dx = \frac{16\sqrt{2}}{3}$$

n = 8 (regular octagon)

$$\text{Area}(A_b) = 2\pi - 2\int_0^1 \left(\sqrt{2-(y-2)^2} - \sqrt{y}\right)dy$$

Q.256

$S=(1+ae,2)$, $s'=(1-ae,2)$

let $(x_0,\ y_0)$ be O.

$OS^2=OP^2 \Rightarrow \boxed{(ae)^2+y_0^2=x^2+(y-y_0)^2}$

given vector restriction gives,

$-aex=x^2+y^2-a^2e^2$

$e^2x^2+(ae)x+a^2(1-2e^2)=0$

let $t=x/a, t\in[-1,1]$

$e^2t^2+et+(1-2e^2)=0$

$\Rightarrow D\ge 0$

$\Rightarrow 8e^2-3\ge 0 \Rightarrow \boxed{e\ge\sqrt{3/8}}$

$e<\dfrac{\sqrt{6}}{4}, \| P \|=0, D<0$

$e=\dfrac{\sqrt{6}}{4}, \| P \|=2, D=0$

$\dfrac{\sqrt{6}}{4}<e\le\dfrac{\sqrt{5}-1}{2}, \| P \|=4, D>0$ in $[-1,1]$ for x.

$e>\dfrac{\sqrt{5}-1}{2}, \| P \|=2, D>0$ but only one x in $[-1,\ 1]$

Here $\| P \|$ is number of points P

Q. 257

$k = 2 \ n(s) = \binom{32}{2} = 496$

If |M|=0, two rows are dependent

So, if one is ϕ, ways to pick ϕ and the other set: $\binom{1}{1} \times \binom{31}{1} = 31$

$$q_2 = \frac{496 - 31}{496} = \frac{15}{16}$$

k=3 $n(s) = \binom{32}{3} = 4960$

Case - 1 empty set, one is \phi and other two from 31

$$\binom{31}{2} = 465$$

Case 2 : $A_1 \cup A_2 = A_3$ where $A_1 \cap A_2 = \phi$ and no set is ϕ.

Let A_3 be of size m and split:

m=2 , $\binom{5}{2} \times 1 = 10$

m=3 , $\binom{5}{3} \times 3 = 30$

$m = 4, \binom{5}{4} \times 7 = 35$

$m = 5, \binom{5}{5} \times 15 = 15$

Total cases = 90

$$p_3 = \frac{4960 - (465 + 90)}{4960} = \frac{881}{992}$$

Idea for a great problem:-

$S = \{1,2,3,4,5\}$

$A = \{1\} \xrightarrow{\text{decode}} \{1,0,0,0,0\} = \vec{e}_1$

$B = \{2\} \xrightarrow{\text{decode}} \{0,1,0,0,0\} = \vec{e}_2$

$\vdots$

$E = \{5\} \xrightarrow{\text{decode}} \{0,0,0,0,1\} = \vec{e}_5$

$\{1,2\} \xrightarrow{\text{decode}} \{1,1,0,0,0\} = \vec{e}_1 + \vec{e}_2$

Let's check $|\{1,2,3\} \cap \{2,3\}| = 2$

Feel! $(\vec{e}_1 + \vec{e}_2 + \vec{e}_3)\cdot(\vec{e}_2 + \vec{e}_3) = 0 + 0 + 1 + 0 + 1 = 2$

(Dot product counts number of elements)

$\boxed{P_{10} = 0, \quad P_{11} = 0, \quad P_k = 0, \quad k > 10}$

In 10 dimension, 5-vectors will always be linearly dependent.

Q.258

$$2f(x)+2f\left(\frac{x-1}{x}\right)=3\tan^{-1}x \qquad \boxed{x\to\frac{x-1}{x}}$$

$$\Rightarrow 2f\left(\frac{x-1}{x}\right)+2f\left(\frac{1}{1-x}\right)=3\tan^{-1}\left(\frac{x-1}{x}\right)\boxed{x\to\frac{1}{1-x}}$$

$$\Rightarrow 2f\left(\frac{1}{1-x}\right)+2f(x)=3\tan^{-1}\left(\frac{1}{1-x}\right)$$

Add (1) + (3) - (2) we get:

$$\frac{4}{3}f(x)=\left(\tan^{-1}x+\tan^{-1}\frac{1}{1-x}-\tan^{-1}\frac{x-1}{x}\right)$$

$x\to 1-x:$

we get $\frac{4}{3}f(1-x)=\left(\tan^{-1}(1-x)+\tan^{-1}\frac{1}{x}-\tan^{-1}\frac{x}{x-1}\right)$

$$\Rightarrow \frac{4}{3}(f(x)+f(1-x))=\frac{3\pi}{2}$$

$$\Rightarrow f(x)+f(1-x)=\frac{9\pi}{8}$$

$$\int_0^1 f(x)dx+\int_0^1 f(1-x)dx=\frac{9\pi}{8}$$

$$\Rightarrow \boldsymbol{\int_0^1 f(x)dx=\frac{9\pi}{16}}$$

Q.259 **Add and Subtract** Evaluating the RHS, we see that it becomes 1 for x < 3 and 9 for x > 3.

By means of integration, we get:

$$|f(x)|+g(x)=\begin{cases}9x+k_0, & \text{if } x>3\\ x+k_1, & \text{if } x<3\end{cases}$$

Obersvation i: f(x) and g(x) can have atmost degree 1 each

Proof: Since f(x) and g(x) are polynomials, they must be continuous functions. The only way for it to evaluate to the piecewise function on RHS, is if

$f(x)<0\forall x>3$ and $f(x)>0\forall x>3$, or vice-versa.

For any arbitrary $n\in\mathbb{N}, n\geq 2$, let a_n and b_n be the coefficients of x^n in f and g respectively.

Since, f takes different signs before and after 3, we have $a_n + b_n$ and $-a_n + b_n$ as the coefficients of x^n in |f(x)| + g(x) (before and after 3, or vice-versa)

Since $n\geq 2$, we have $a_n+b_n=0$ and $-a_n+b_n=0$ since no x^n terms are on RHS for $n\geq 2$.

Therefore, $a_n = 0$, $b_n = 0$.

Since f is linear, and |f(x)| is non-differentiable at 3. We get f has a zero at 3, so f(x) = a(x - 3) as the only solution.

We take g(x) = bx + c for the other linear polynomial.

WLOG, assume a > 0. -a(x - 3) will be the only other solution to f(x), since |y| = |-y|.

$$|a(x-3)|+(bx+c)=\begin{cases}a(x-3)+(bx+c)=(b+a)x+(c-3a), & \text{if } x>3\\ -a(x-3)+bx+c=(b-a)x+(c+3a) & \text{if } x<3\end{cases}$$

Comparing coefficients, we get b + a = 9, b - a = 1, therefore b = 5, a = 4

We also have, g(0) = b(0) + c = c = 12.

Therefore, the polynomials are g(x) = 5x + 12 and $f(x)=\pm 4(x-3)$.

Q.260

$a_{2026-i} = 1 - a_i$ and

$1 - 3a_i + 3a_i^2 = (1 - 3a_i + 3a_i^2 - a_i^3) + a_i^3$

$= (1 - a_i)^3 + a_i^3 = a_i^3 + a_{2026-i}^3$

Replace $i \to 2026 - i$ in SumS

$$2S = S + S = \sum \frac{a_i^3}{a_i^3 + a_{2026-i}^3} + \sum \frac{a_{2026-i}^3}{a_i^3 + a_{2026-i}^3}$$

$= 2027$

$$\Rightarrow \mathbf{S = \frac{2027}{2}}$$

Q.261

$$S = \sum_{n=1}^{\infty} \frac{1}{(4n+3)(4n+5)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{4n+3} - \frac{1}{4n+5}\right)$$

$$\Rightarrow S = \frac{1}{2}\left(1/7 - 1/9 + 1/11 - 1/13 + \ldots\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(\int_0^1 (x^{4n+2} - x^{4n+4})dx\right) \text{ (Not working Mass Killing!)}$$

$$= \frac{1}{2}\int_0^1 \sum_{n=1}^{\infty} x^{4n+2}(1 - x^2)dx$$

$$= \frac{1}{2}\int_0^1 (1 - x^2)\frac{x^6}{1 - x^4}dx$$

$$= \frac{1}{2}\int_0^1 \frac{x^6}{1 + x^2}dx$$

$$\lim_{n\to\infty} T_n = \int_0^1 \frac{x^6\,dx}{1 + x^2}$$

Extra: $\int_0^1 \frac{x^6}{1 + x^2}dx = \int_0^1 \left(x^4 - x^2 + 1 - \frac{1}{1 + x^2}\right)dx$

$$= \frac{1}{2}\left(\frac{13}{15} - \frac{\pi}{4}\right)$$

OMG: Natural numbers give π Hahaaa!

Q.262

$64 \in g_{a,r}$, $a, r \in \mathbb{N}$, $r > 1$

$64 = 2^6$ both a, r must be powers of 2.

let $r = 2^k, k \in \{1,2,3,4,5,6\}$

$r = 2,\ a = \dfrac{64}{r^n}$, $a \in \{2^6, 2^5, 2^4, 2^3, 2^2, 2^1\}$

$r = 4,\ a \in \{2^6, 2^4, 2^2, 1\}$

$r = 8,\ a \in \{2^6, 2^3, 1\}$

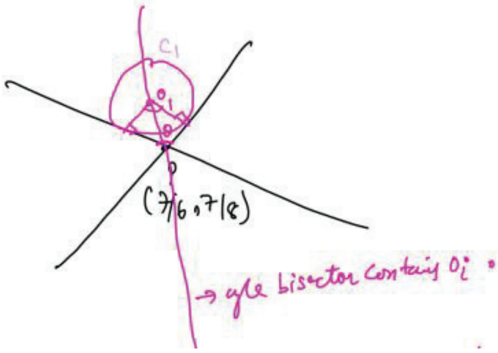
$r = 16,\ a \in \{2^6, 2^2\}$

$r = 32,\ a \in \{2^6, 2\}$

$r = 64,\ a \in \{2^6, 2^0\}$

Total unique sequence = 19

Q.263



Horizontal

$$d = \frac{r}{\sin\theta}$$

$$\tan\theta = 3/4$$

$$\Rightarrow d_h = \frac{5}{6},\ O_1, O_2 \equiv (2, 7/8), (1/3, 7/8)$$

Vertical

$$\tan\theta = 4/3 \Rightarrow \sin\theta = 4/5$$

$$d_v = 5/8$$

$$O_3, O_4 \equiv (7/6, 3/2) \text{ and } (7/6, 1/4)$$

$$\sum x_i = \frac{14}{3}$$

$$\max(\text{Area}) = \frac{1}{2} \times \frac{1/2}{3/4} = 1/3$$

$$\min(\text{Area}) = \frac{3}{16}$$

Q.264

$$s_n = \frac{n-1}{2}$$

$$S = \sum_{r=0}^{n} \frac{(-1)^r \binom{n}{r}^2}{r+1}$$

$$= \frac{1}{n+1} \sum (-1)^r \binom{n+1}{n-r} \binom{n}{r}$$

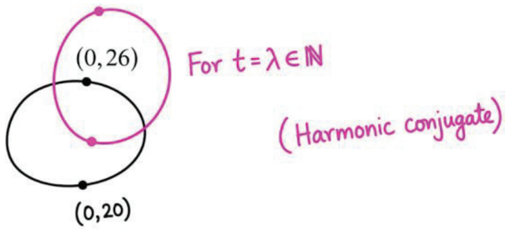
Coe of x^n in $(1+x)^{n+1}(1-x)^n$

OR

Vandermondes!

gives k = 50

Q. 265 Blind way:



$$\frac{PA}{PB}=t \Rightarrow (t^2-1)x^2+(t^2-1)y^2$$

$$-(52t^2-40)y+(26^2t^2-20^2)=0$$

$$g_1=0, \quad f_1=\frac{20-26n^2}{n^2-1}, \quad c_1=\frac{676n^2-400}{n^2-1}$$

eqn of S_c

$$x^2+y^2-2gx-46y+520=0$$

(as centre is (g, 23))

$$g_2=-g$$

$$f_2=-23$$

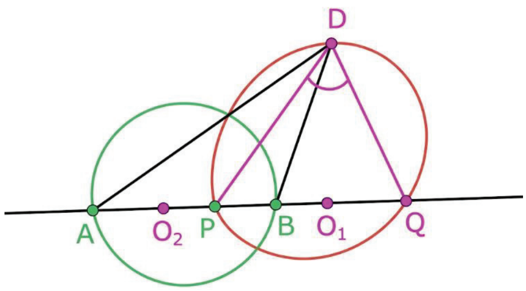
$$c_2=520$$

Use $2g_1g_2+2f_1f_2=c_1+c_2$

we get L.H.S = R.H.S (Nice)

All are orthogonal!

Geometrically

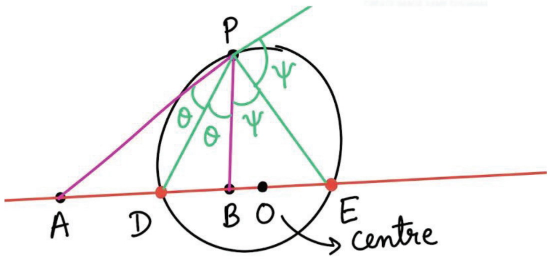


$$O_1A \cdot O_1B = r_1^2 \text{ (similarity)}$$

$$\text{But } O_1A \cdot O_1B = O_1O_2^2 - r_2^2 \text{ (power of point)}$$

$\Rightarrow \boxed{\mathbf{r_1^2 + r_2^2 = O_1O_2^2}} \Rightarrow$ orthogonal

Proof:



$$\text{In } \Delta OPE:$$

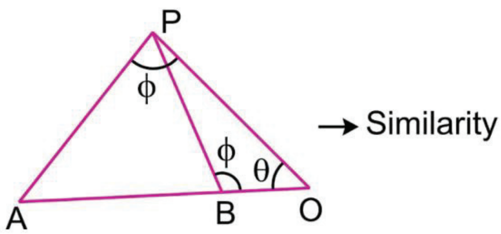
$OE = OP = R$

$\angle OPE = \angle OEP$ (as $OP = OE$)

$\angle OBP = \angle OPA$ (angle bisector chase)

$\triangle OBP \sim \triangle OPA$

$$\Rightarrow \frac{OB}{OP} = \frac{OP}{OA} \Rightarrow \boxed{OA \cdot OB = OP^2}$$



(Note: No two circles of Apollonius can intersect, else at a point two ratios.)

Q.266

$$x^2 - y = k\pi$$
$$|x| + |y| = 2\pi \text{ for } k \in Z$$

for $k \le -3 \to 0$ intersection (check vertex)

$k = -2 \to 1$ intersection

$-1 \le k \le 1 \to 2$ intersections

$k = 2, \to 5$ intersections

$k \ge 13, \to 0$ intersections

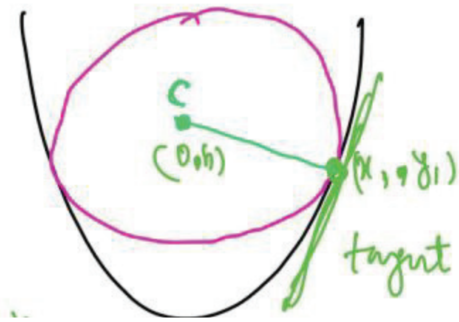
1 + 2 × 3 + 5 + 10 × 4 = 52

Q.267

$$f(x) = \frac{(x^5 - 1) + (x^5 - x) + \cdots + (x^5 - x^4)}{(x - 1)}$$
$$= \frac{5x^6 - 6x^5 + 1}{(x - 1)^2}$$

Denominator becomes $(1 - \omega)(1 - \omega^2)(1 - \omega^3)(1 - \omega^4) = 5$

Q.268



$$y - y_1 = \frac{-5}{x_1}(x - x_1) \quad ((0,h) \text{ lies on it})$$

$$\Rightarrow h = y_1 + 5$$

$$r^2 = x_1^2 + (y_1 - h)^2 = 10y_1 + 25$$

$$h = \frac{r^2 + 25}{10}$$

$$\text{Since, } r_1 = 13 \Rightarrow h = \frac{194}{10}$$

For C_2 and C_1 be externally touching:

$$\left|\frac{r_2^2 + 25}{10} - \frac{194}{10}\right| = 13 + r_2$$

upper circle: $h_2 > h_1$

$$\Rightarrow r_2^2 - 10r_2 - 299 = 0$$

$$\Rightarrow r_2 = 23, \quad r_2 \neq -13$$

lower circle: $h_2 < h_1$

$$\Rightarrow r_2^2 + 10r_2 - 39 = 0$$

$$\Rightarrow (r_2 - 3)(r_2 + 13) = 0$$

$$\Rightarrow r_2 = 3 \quad \text{(ghost solution)},$$

$$r_2 \neq -13$$

(If r=3 maths say y=-1.6 if parabola existed

Q.269

Exponential generating fn:

Digit 1

$$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots=\frac{e^x+e^{-x}}{2}$$

Digit 2

$$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots=\frac{e^x-e^{-x}}{2}$$

Digit 3 (atleast once)

$$\frac{x}{1!}+\frac{x^2}{2!}+\ldots=e^x-1$$

Digit 4

$$1+x+\frac{x^2}{2!}+\ldots=e^x$$

Digit 5

$$1+x+\frac{x^2}{2!}+\ldots=e^x$$

[As no restrictions]

$$g(x)=\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right)(e^x-1)(e^x)^2$$

$$\Rightarrow g(x)=\frac{1}{4}(e^{4x}-1)(e^x-1)$$

$$\boxed{a_n=\text{Coe of }\frac{x^n}{n!}\text{ in } g(x)}$$

$$\boxed{a_n=\frac{5^n-4^n-1}{4}}$$

Q.270

f(x) is odd function, hence symmetric about origin. So for every α there exists β such that $\alpha+\beta=0$. Further if the maximum value is attained at a, b, c, $\ldots$ then minimum must be attained at -a, -b, -c. Hence sets have equal elements.

Q.271

$\frac{\Delta_L}{\Delta_U}=1/2$ (parabola standard property)

Q.272

Take $\int_0^{\pi} f_2(t)dt = c$

$\Rightarrow \boxed{f_2(x) = 1 + \frac{2c}{\pi}\cos^2 x} \rightarrow$ we get

$\int_0^{\pi}\left(1 + \frac{2c}{\pi}\cos^2 x\right)dx = c \Rightarrow$ Inconsistency.

Hence $\| f_2(x) \| = 0$.

Similarly others can be done.

$\| f_1(x) \|$ is

Let $\int_0^1 t^2 f_1(t)dt = A$

$\Rightarrow f_1(x) = \frac{7x}{8} + \frac{Ax}{2}$

$\Rightarrow \int_0^1 t\left(\frac{7t}{8} + \frac{At}{2}\right)^2 dt = A$

$\Rightarrow A = 1/4, 49/9.$

Hence $f_1(x) = x$ or $7x$.

Q.273

$\cos x \cos 2x \cos 3x \cos 4x \cos 5x$

$= \cos((1 \pm 2 \pm 3 \pm 4 \pm 5)x)$ (Taking all possible combinations)

Since $\int_0^{\pi} \cos(nx)dx = \begin{cases} 0, & n \neq 0 \\ \pi, & n = 0 \end{cases}$ for $n \in \mathbb{N}$.

So we are seeking for combination of $(1 \pm 2 \pm 3 \pm \cdots)$ which never becomes zero.

Q.274

$f(x) = 2\left(\frac{1}{\sin 2x} - 1\right).$

In $(-\pi/2, 0)$, $f'(x) = -4\operatorname{cosec} 2x \cot 2x$

$f'(x) = 0$ at $x = -\pi/4$ gives $b = -4$.

and $a = -\pi/4$

$a^2 + b^2 = \frac{\pi^2}{16} + 16$

Rest use $(g(x)) = g(f(x)) = x$.

Q.275 (D)

$K^2\vec{r} = \hat{n}\times(\hat{n}\times\vec{r})$ (as K represents $\hat{n}\times\ldots$)

$= \hat{n}(\hat{n}\cdot\vec{r}) - \vec{r}(\hat{n}\cdot\hat{n})$

$= \hat{n}(\hat{n}\cdot\vec{r}) - \vec{r}\cdot I = \hat{n}\hat{n}^T\vec{r} - \vec{r}I$

Thus $K^2 = \hat{n}\hat{n}^T - I$ as an operator)

(A)

$|M - I| = |M - MM^T| = |I - M^T|$

$= |I^T - M^T| = |(I - M)^T| = |I - M|$

$\Rightarrow \quad |M - I| = (-1)^3 |M - I| \Rightarrow |M - I| = 0$

Alternate : $MM^T = I$ and $|M| = 1$

preserves length and angle (Just rotation)

So, $MX = \lambda X, \quad X \neq 0$

$\Rightarrow X^T M^T = \lambda X^T$

$\Rightarrow X^T M^T M X = \lambda^2 X^T X$

$\Rightarrow X^T X = \lambda^2 X^T X$

$\Rightarrow (\lambda^2 - 1)\underbrace{X^T X}_{(\neq 0)} = 0$

$\Rightarrow \lambda = \pm 1$

Since, $|M| = \lambda_1\lambda_2\lambda_3$ (eigen values)

For orthogonal matrix $|\lambda| = 1$ (Simple to feel!)

$\lambda_1\lambda_2\lambda_3 = 1$(given)

$$\left.\begin{array}{c} 1\cdot(-1)\cdot(-1) = 1 \\ 1\cdot 1\cdot 1 = 1 \\ 1\cdot e^{i\theta}\cdot e^{-i\theta} = 1 \end{array}\right\}\text{only possibility}$$

So, 1 is always an eigenvalue.

Hence, For all $M \in S, |M - I| = 0$

$|M + I|$ may or may not be zero.

$|M^5 - I| = |M - I||M^4 + M^3 + M^2 + M + I| = 0$

(D)

$kr = \hat{n} \times \vec{r}$(vector way!)

$k^2 r = \hat{n} \times (\hat{n} \times \vec{r})$

$k^3 r = \hat{n} \times (\hat{n} \times (\hat{n} \times \vec{r}))$

$k^2 r = \hat{n}(\hat{n} \cdot \vec{r}) - \vec{r}(\hat{n} \cdot \hat{n})$

$= \hat{n}(\hat{n} \cdot \vec{r}) - \vec{r}$

$\boxed{k^2 r = \hat{n}(\hat{n} \cdot \vec{r}) - r}$

$\boxed{k^3 r = -kr}$

For skew symmetric of odd order

$|A| = 0 = \lambda_1 \lambda_2 \lambda_3$

$\lambda_1 = 0$, $\lambda_2, \lambda_3 = \pm i$ $\leftarrow$ purely imaginary

$\lambda(\lambda - i)(\lambda + i) = 0 \Rightarrow \lambda^3 + \lambda = 0$

Caley – Hamiltonforces $A^3 + A = 0$

Q.276

$$x^2 f'(x) - 4xf(x) = \frac{-3x^4(1-x)^4}{1+x^2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{f(x)}{x^4}\right) = \frac{-3(1-x)^4}{1+x^2}$$

$$\Rightarrow \left.\frac{f(t)}{t^4}\right|_x^1 = -3\int_x^1 \frac{(1-t)^4}{1+t^2}dt$$

$$\Rightarrow f(1) - L = -3\int_0^1 \frac{(1-t)^4}{1+t^2}dt$$

As $x \to 0^+, \frac{f(x)}{x^4} = L$ (so we need n=4 for non-zero)

Since

$$\int_0^1 \frac{(1-t)^4}{1+t^2} = \int_0^1 \left((t^2 - 4t + 5) - \frac{4}{1+t^2}\right) = \frac{10}{3} - \pi$$

$$\Rightarrow \left(\frac{22}{7} - \pi\right) - L = -3\left(\frac{10}{3} - \pi\right)$$

$$\Rightarrow \boxed{L = \frac{92}{7} - 4\pi}$$

Q.277

$$\sum_{n=1}^{\infty} \frac{1}{k^n} = \frac{1}{k-1}$$

$$\sum_{n=1}^{\infty} \frac{n}{k^n} = \frac{k}{(k-1)^2}$$

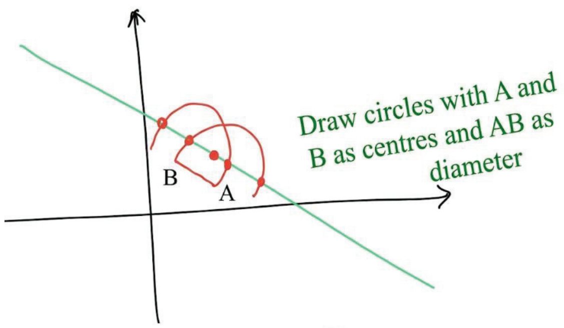
$$\sum_{n=1}^{\infty} \frac{n^2}{k^n} = \frac{k(k+1)}{(k-1)^3}$$

For k=2, 6a + 2b + c = 2026

k=3, 6a + 3b + 2c = 8104

b + c = 8104 - 2026 = 6078

Q.278



Q.279

$\alpha\beta\gamma = 7(\alpha+\beta+\gamma) \quad , \quad \alpha,\beta,\gamma \in \mathbb{R}^+$

$\Rightarrow$ let $x = \sqrt{7}\tan\theta$ then $\sum \tan\theta_i = \prod \tan\theta_i$

$\Rightarrow \sum \theta_i = \pi$, given ratio $k = n^2 \sin^2\theta$,

We have

$$\sin\theta_1 = \frac{\sqrt{k}}{15}$$

$$\sin\theta_2 = \frac{\sqrt{k}}{12}$$

$$\sin\theta_3 = \frac{\sqrt{k}}{10}$$

Solving the triangle,

$$\sin\theta_1 = \sin\theta_2\cos\theta_3 + \sin\theta_3\cos\theta_2$$

$$\Rightarrow 8 = \sqrt{100-k} + \sqrt{144-k} \Rightarrow \boxed{k = \frac{1575}{16}}$$

Substitute $\alpha = \sqrt{7}\tan\theta_1$

$$\sin^2\theta_1 = \frac{7}{16} \Rightarrow \tan^2\theta_1 = \frac{7}{9} \Rightarrow \alpha = \frac{7}{3}$$

$$\text{Roots}: \left\{\frac{7}{3}, \frac{35}{9}, 21\right\}$$

Hence 245p = 9 and 15q = 77

Q.280

$$g\left(\frac{\pi}{12}\right)=\int_{\pi/12}^{\pi/4}\cot^{-1}(2-\tan x)\,dx$$

$$g\left(\frac{\pi}{3}\right)=\int_{\pi/4}^{\pi/3}\cot^{-1}(2-\tan x)\,dx$$

let $\cot^{-1}(2-\tan x)=t$

$\Rightarrow 2-\tan x=\cot t$

at $x=\pi/4\rightarrow t=\pi/4$

at $x=\pi/3\rightarrow 2-\sqrt{3}=\cot t$

$\Rightarrow \tan t=2+\sqrt{3}\Rightarrow \boxed{t=\frac{5\pi}{12}}$

$$g\left(\frac{\pi}{3}\right)=\int_{\pi/4}^{5\pi/12}\frac{t\csc^2 t\,dt}{1+(2-\cot t)^2}$$

$\tan x=2-\cot t$

$\Rightarrow \sec^2 x\,dx=\csc^2 t\,dt$

$$dx=\frac{\csc^2 t\,dt}{1+(2-\cot t)^2}$$

Similarity $g\left(\frac{\pi}{12}\right)=\int_{\pi/12}^{\pi/4}\cot^{-1}(2-\tan x)\,dx$

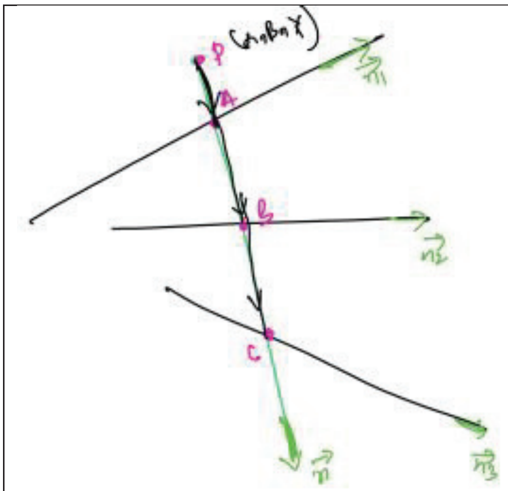
$2-\tan x=\cot\theta$

So, $\frac{\pi}{12}\rightarrow\frac{\pi}{6}$

$\frac{\pi}{4}\rightarrow\frac{\pi}{4}$

$$g\left(\frac{\pi}{12}\right)=\int_{\pi/6}^{\pi/4}\frac{t\csc^2 t\,dt}{1+(2-\cot t)^2}$$

Q.281



$$(\overrightarrow{PA} \times \overrightarrow{n_1}) \cdot \vec{n} = 0$$

$$(\overrightarrow{PB} \times \overrightarrow{n_2}) \cdot \vec{n} = 0$$

$$(\overrightarrow{PC} \times \overrightarrow{n_3}) \cdot \vec{n} = 0$$

$$\Rightarrow \quad [\overrightarrow{PA} \times \overrightarrow{n_1} \quad \overrightarrow{PB} \times \overrightarrow{n_2} \quad \overrightarrow{PC} \times \overrightarrow{n_3}] = 0$$

Q.282

$$f'(c) = \frac{f(3) - f(2)}{3 - 2}, \quad c \in (2,3)$$

Since, $f'(c) \geq \pi$ (super easy!)

Trap! You will think of differential equation.

Q.283

$$\sum_{k=0}^{14}\binom{14+k}{14}\binom{20-k}{10}$$

$\downarrow$

$$\binom{20-k}{10-k}$$

$$(1-x)^{-n}=\sum_{r=0}^{\infty}\binom{n+r-1}{r}x^{r},\quad n\in\mathbb{N}\ \text{ for } |x|<1.$$

For $n=15,\binom{14+k}{k}$ is $[x^{k}]$ in $(1-x)^{-15}$

and $n=1,\binom{11+(10-k)-1}{10-k}=\binom{20-k}{10-k}$

$[x^{10-k}]$ in $(1-x)^{-11}$.

$\uparrow$ Coefficient of

So, S is $[x^{10}]$ in $(1-x)^{-15}\cdot(1-x)^{-11}=(1-x)^{-26}$

So, $[x^{10}]$ in $(1-x)^{-26}=\binom{35}{10}=\binom{35}{25}$

$\min|n-r|=|35-25|=10$